Quantum Langevin equation

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From the viewpoint of Schrödinger, we have studied a hormonic oscillator coupled to an Ohmic environment and derive the resultant master equation for the damped harmonic oscillator. We shall now reexamine the damped harmonic oscillator from the viewpoint of Heisenberg and discuss the resultant *quantum Heisenberg-Langevin equation*.

I. REMARKS

Bofore going to the issue of treating damped harmonic oscillators from the viewpoint of Heisenberg, let us pause for a moment and streamline what we have learned by connecting them with some important concepts.

A. Fermi's golden rule [1]

Let us consider again the situation in which an LC circuit capacitively coupled to a 1D transmission line. The interaction Hamiltonian is then given by (see Eq. (48) in *note 2016-11-28*)

$$\begin{aligned} H_i &= Q_s(t)V(t) \\ &= \underbrace{\sqrt{\frac{\hbar}{2L_0\omega_0}}\left(\hat{a}+\hat{a}^\dagger\right)}_{Q_s(t)}\underbrace{\left(V^{\rightarrow}(0,t)+V^{\leftarrow}(0,t)\right)}_{V(x=0,t)} \\ &= \sqrt{\frac{\hbar}{2L_0\omega_0}}\left(\hat{a}+\hat{a}^\dagger\right)\left(-2i\int_0^\infty \frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)-\hat{c}^\dagger(\omega)\right)\right), \end{aligned} \tag{1}$$

where the factor "2" in the second parenthesis in the last line coming from the fact that $V^{\rightarrow}(0,t) = V^{\leftarrow}(0,t)$ for the open terminal at $x = 0$. Now we invoke the *secular approximatin*, with which only the terms varing slowly with respect to the *coarse-grained time* Δt are retained, to obtain

$$\begin{aligned} H_i &= -i\hbar\sqrt{\frac{Z_p}{L_0}}\int_0^\infty \frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega)-\hat{a}\hat{c}^\dagger(\omega)\right) \\ &= -i\hbar\sqrt{\frac{Z_p}{L_0}}\int_{-\infty}^\infty \frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega)-\hat{a}\hat{c}^\dagger(\omega)\right), \end{aligned} \tag{2}$$

where the second equation is due to the fact that the integrand is only non-zero around $\omega \sim \omega_0$ so that the domain of integration can be extended down to $-\infty$. This Hamiltonian is indeed the one we used in Eq. (4) in *note 2016-12-05* with the coupling rate $f(\omega)$ being assumed to be freqency-independent (white), that is,

$$f(\omega) = f^*(\omega) = \sqrt{\frac{Z_p}{L_0}} = \sqrt{\Gamma}. \tag{3}$$

This establishes the connection between the coupling constant $f(\omega)$ and the Einstein-A-coefficient like spontaneous emission rate Γ. Indeed, we have already encontered this, that is, Eq. (44) in *note 2016-12-05* says

$$\Gamma = |f(\omega_0)|^2 = \int_{-\infty}^\infty d\omega\,|f(\omega)|^2\,\delta(\omega-\omega_0), \tag{4}$$

which means that the Einstein-A-coefficient like spontaneous emission rate Γ can be obtained from the square of the coupling rate. This is called *Fermi's golden rule*.

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B. Detailed balance [1]

We then identify

$$\begin{aligned}\Gamma_\downarrow = \Gamma' + \Gamma &= |f(\omega_0)|^2 \left(\langle n(\omega_0)\rangle + 1\right) \\ &= \frac{Z_p}{L_0}\left(\langle n(\omega_0)\rangle + 1\right) \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(\underbrace{2Z_p\hbar\omega_0\left(\langle n(\omega_0)\rangle + 1\right)}_{S_{VV}(\omega_0)}\right) \\ &= \frac{1}{2\hbar\omega_0 L_0} S_{VV}(\omega_0),\end{aligned} \tag{5}$$

and

$$\begin{aligned}\Gamma_\uparrow = \Gamma' &= |f(\omega_0)|^2 \langle n(\omega_0)\rangle \\ &= \frac{Z_p}{L_0}\langle n(\omega_0)\rangle \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(\underbrace{2Z_p\hbar\omega_0\langle n(\omega_0)\rangle}_{S_{VV}(-\omega_0)}\right) \\ &= \frac{1}{2\hbar\omega_0 L_0} S_{VV}(-\omega_0).\end{aligned} \tag{6}$$

In words, the downward decay rate $\Gamma_\downarrow$ and the upward decay rate $\Gamma_\uparrow$ are related to the noise spectral densities $S_{VV}(\omega_0)$ and $S_{VV}(-\omega_0)$, respectively.

Now the rate equation (see Eq. (50) in *note 2016-12-05*) suggests that

$$\begin{aligned}\frac{d}{dt}\langle N\rangle &= \Gamma_\uparrow\langle N+1\rangle - \Gamma_\downarrow\langle N\rangle \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(S_{VV}(-\omega_0)\langle N+1\rangle - S_{VV}(-\omega_0)\langle N\rangle\right).\end{aligned} \tag{7}$$

Note that N stands for the photon number in the LC circuit and $n(\omega_0)$ stands for the photon number in the 1D transmission line at angular frequency ω_0. In the steady state we have $\frac{d}{dt}\langle N\rangle = 0$ and thus

$$\begin{aligned}\frac{\Gamma_\uparrow}{\Gamma_\downarrow} &= \frac{S_{VV}(-\omega_0)}{S_{VV}(\omega_0)} \\ &= \frac{\langle N\rangle}{\langle N+1\rangle} = \frac{\frac{1}{e^{\frac{\hbar\omega_0}{k_B T}}-1}}{\frac{1}{e^{\frac{\hbar\omega_0}{k_B T}}-1}+1} = e^{-\frac{\hbar\omega_0}{k_B T}},\end{aligned} \tag{8}$$

where the thermal equilibrium is assumed. Equation (8) is called the *detailed balance condtion*, suggesting that the asymmetry of the noise power spectral densities $S_{VV}(-\omega_0)$ and $S_{VV}(\omega_0)$ are related to the temperature T. This fact was used in Eq. (60) in *note 2016-11-28* when we deduce the classical Nyquist formula from the quantum counterpart.

II. QUANTUM LANGEVIN EQUATION

A. Classical Langevin equation [1, 2]

Let us study the situation in which an LC circuit *system* (a harmonic oscillator) coupled to a transmission line *bath* (a boson field) characterized by the impedance Z_p. The *Langevin equation* for the LC circuit is obtained by the following argument. Remembering that the right-moving voltage and the right-moving current are related as

$$\begin{aligned}\frac{\partial}{\partial x}V^{\rightarrow}(x,t) &= \frac{\partial}{\partial x}\dot{\varphi}^{\rightarrow}(x,t) \\ &= \frac{\partial}{\partial t}\underbrace{\frac{\partial}{\partial x}\varphi^{\rightarrow}(x,t)}_{-lI^{\rightarrow}(x,t)} = -l\left(\frac{\partial}{\partial t}I^{\rightarrow}(x,t)\right). \end{aligned} \tag{9}$$

Thus we have the current from the following expression:

$$I^{\rightarrow}(x,t) = -\frac{1}{l}\int_{-\infty}^{t} d\tau \left(\frac{\partial}{\partial x}V^{\rightarrow}(x,\tau)\right). \tag{10}$$

By plugging

$$V^{\rightarrow}(x,t) = -i\int_0^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(kx-\omega t)} - h.c.\right) \tag{11}$$

$$V^{\leftarrow}(x,t) = -i\int_0^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(-kx-\omega t)} - h.c.\right) \tag{12}$$

into the *constitutive equation* (10), which is essentially the Newton's law for the transmission line, we have

$$I^{\rightarrow}(x,t) = \frac{V^{\rightarrow}(x,t)}{Z_p} \tag{13}$$

$$I^{\leftarrow}(x,t) = -\frac{V^{\leftarrow}(x,t)}{Z_p}. \tag{14}$$

Since the boundary between the transmission line bath and the LC circuit system at $x = 0$ is open we have

$$V(x=0,t) = V^{\rightarrow}(x=0,t) + V^{\leftarrow}(x=0,t) \equiv V_{out}(t) + V_{in}(t) \tag{15}$$

$$\begin{aligned} I(x=0,t) &= I^{\rightarrow}(x=0,t) + I^{\leftarrow}(x=0,t) \\ &= \frac{1}{Z_p}\left(V^{\rightarrow}(x=0,t) - V^{\leftarrow}(x=0,t)\right) \equiv \frac{1}{Z_p}\left(V_{out}(t) - V_{in}(t)\right). \end{aligned} \tag{16}$$

This can be considered as the *classical input-output relation.* By eliminating $V_{out}(t)$ the voltage and the current relation at the boundary becomes

$$V(x=0,t) = Z_p I(x=0,t) + 2V_{in}(t). \tag{17}$$

Now let us consider the following LCR circuit equation, where the resistance stems from the coupling to the semi-infinite transmission line bath characterized by the impedance Z_p. By Kirchhoff 's law we have

$$\frac{Q(t)}{C_0} + L_0\dot{I}(x=0,t) + V(x=0,t) = 0. \tag{18}$$

With the emf voltage $V(x=0,t)$ due to the semi-infinite transmission line bath, which is given by Eq. (17), the circuit equation becomes

$$\frac{Q(t)}{C_0} + Z_p I(x=0,t) + L_0\dot{I}(x=0,t) = -2V_{in}(t), \tag{19}$$

which leads to the following *white-noise-form* Langevin equation:

$$\ddot{Q}(t) + \underbrace{\Gamma}_{\frac{Z_p}{L_0}}\dot{Q}(t) + \underbrace{\omega_0}_{\frac{1}{L_0C_0}}Q(t) = -\frac{2V_{in}(t)}{L_0}, \tag{20}$$

where $V_{in}(t)$ and $\dot{Q}(t)$ are the *stochastic variables*, called

$$\begin{aligned} V_{in}(t) &: \text{Wiener (white noise) process} \\ \dot{Q}(t) = I(t) &: \text{Ornstein} - \text{Uhlenbeck process} \end{aligned}$$

respectively. Here the averaged values of $V_{in}(t)$ exhibits *strange* traits [2]:

$$\begin{aligned} \langle V_{in}(t)\rangle &= 0 & (21)\\ \langle V_{in}(t)V_{in}(0)\rangle &= \delta(t)S^{\leftarrow}_{VV}, & (22)\end{aligned}$$

where the spectral density $\bar{S}^{\leftarrow}_{VV}(\omega)$ is given by

$$\bar{S}^{\leftarrow}_{VV}(\omega) = \frac{1}{4}\bar{S}_{VV}(\omega) = Z_p\hbar\omega\left(n(\omega)+\frac{1}{2}\right). \tag{23}$$

The above Langevin equation, Eq. (20) is a typical example of the *stochastic differential equation*, for which the more careful mathematical manipulation is required than for the ordinary differential equation [2]. Nevertheless, we shall *abuse* the Fourier transform and get

$$Q(\omega) = \frac{1}{(\omega_0^2-\omega^2)-i\omega\Gamma}\left(-\frac{2V_{in}(\omega)}{L_0}\right), \tag{24}$$

which nevertheless gives us the *correct* spectral density

$$S_{QQ}(\omega) = \frac{1}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2}\left(\frac{4\bar{S}^{\leftarrow}_{VV}(\omega)}{L_0^2}\right). \tag{25}$$

From the *virial theorem* the capacitive energy $\langle\frac{Q^2}{2C_0}\rangle$ and inductive energy $\langle\frac{\varphi^2}{2L_0}\rangle$ share the same energy $\frac{E}{2}$. We thus have the following energy spectral density for the LCR circuit:

$$\begin{aligned} S_E(\omega) &= 2\frac{S_{QQ}(\omega)}{2C_0} = \frac{1}{C_0}\frac{1}{\left(\omega_0^2-\omega^2\right)^2+\omega^2\Gamma^2}\left(\frac{4Z_p\hbar\omega}{L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right)\\ &\sim \frac{1}{4\omega_0^2\left(\omega_0-\omega\right)^2+\omega_0^2\Gamma^2}\left(\frac{4Z_p\hbar\omega}{C_0L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right)\\ &= \frac{1}{\left(\omega_0-\omega\right)^2+\frac{\Gamma^2}{4}}\left(\frac{Z_p\hbar\omega}{\omega_0^2C_0L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right)\\ &= \frac{1}{\left(\omega_0-\omega\right)^2+\frac{\Gamma^2}{4}}\left(\frac{Z_p}{L_0}\hbar\omega\left(n(\omega)+\frac{1}{2}\right)\right)\\ &= \frac{\Gamma}{\left(\omega_0-\omega\right)^2+\frac{\Gamma^2}{4}}\left(\hbar\omega\left(n(\omega)+\frac{1}{2}\right)\right). \end{aligned} \tag{26}$$

We shall be led to the same energy spectral density when we use the more general quantum Heisenberg-Langevin approach we shall now learn. For the explicit derivation, try Problem III B.

B. Quantum Heisenberg-Langevin equation [1, 3]

Let us reexamine the LCR circuit quantum mechanically with the Heisenberg picture. We shall assume the total Hamiltonian to be

$$H = H_s + H_b + H_i, \tag{27}$$

where H_s, H_b, and H_i are the Hamiltonians of the LC circuit (the system), the transmission line (the Ohmic environment), which are respectively given by

$$\begin{aligned} H_s &= \hbar\omega_0\hat{a}^\dagger\hat{a} & (28)\\ H_b &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hbar\omega\hat{c}^\dagger(\omega)\hat{c}(\omega) & (29)\end{aligned}$$

and the interaction Hamiltonian is given by Eq. (2). The Heisenberg equation of motion for the environment is

$$\dot{\hat{c}}(\omega,t) = \frac{i}{\hbar}\left[H,\hat{c}(\omega,t)\right] = -i\omega\hat{c}(\omega,t)+\sqrt{\Gamma}\hat{a}(t). \tag{30}$$

We can find the formal solution of Eq. (30) as

$$\hat{c}(\omega,t) = e^{-i\omega(t-t_0)}\hat{c}(\omega,t_0) + \sqrt{\Gamma}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau). \tag{31}$$

The Heisenberg equation of motion for the system, on the other hand, is given by

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}\left[H,\hat{a}(t)\right] = -i\omega_0\hat{a}(t) - \sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega,t). \tag{32}$$

Now let us define the *slowly-varing variable* $\hat{\alpha}(t)$ as $\hat{a}(t) = \hat{\alpha}(t)e^{-i\omega_0 t}$. Plugging this $\hat{\alpha}(t)$ in Eq. (32) we can eliminate the first term:

$$\dot{\hat{\alpha}}(t) = -\sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega,t)e^{i\omega_0 t}. \tag{33}$$

By plugging the solution for $\hat{c}(\omega,t)$ in Eq. (31) into Eq. (33) we have

$$\begin{aligned}
\dot{\hat{\alpha}}(t) &= -\sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i(\omega_0-\omega)t}e^{i\omega t_0}\hat{c}(\omega,t_0) - \Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{t_0}^{t}d\tau e^{i(\omega_0-\omega)(t-\tau)}\hat{\alpha}(\tau)\\
&= \underbrace{-\sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i(\omega_0-\omega)t}\hat{c}(\omega,t_0)e^{i\omega t_0}}_{\hat{c}(t)} - \Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{\alpha}(t)\int_{0}^{t-t_0}d\tau' e^{i(\omega_0-\omega)\tau'}\\
&= -\sqrt{\Gamma}\hat{c}(t) - \Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{\alpha}(t)\int_{0}^{\infty}d\tau' e^{i(\omega_0-\omega)\tau'}\\
&= -\sqrt{\Gamma}\hat{c}(t) - i\Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{\alpha}(t)\left(-i\lim_{\epsilon\to 0^+}\int_{0}^{\infty}d\tau e^{i[(\omega_0-\omega)+i\epsilon]\tau}\right)\\
&= -\sqrt{\Gamma}\hat{c}(t) - i\Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{\alpha}(t)\underbrace{\left(\lim_{\epsilon\to 0^+}\frac{1}{(\omega_0-\omega)+i\epsilon}\right)}_{\frac{\mathcal{P}}{\omega_0-\omega}-i\pi\delta(\omega_0-\omega)}\\
&= -\sqrt{\Gamma}\hat{c}(t) - i\underbrace{\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Gamma}{\omega_0-\omega}}_{\Delta}\hat{\alpha}(t) - \underbrace{\Gamma\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\pi\delta(\omega_0-\omega)}_{\frac{\Gamma}{2}}\hat{\alpha}(t)\\
&= -\sqrt{\Gamma}\hat{c}(t) - \left(i\Delta + \frac{\Gamma}{2}\right)\hat{\alpha}(t),
\end{aligned} \tag{34}$$

where we put $\tau' = t-\tau$. Note that 1) since the variation of the slowly-varing variable $\hat{\alpha}(\tau)$ can be assumed to be constant, $\hat{\alpha}(t)$, and thus put outside of the integration with respect to τ', 2) since $t_0 \to -\infty$ the domain of integration of τ' can be extended to ∞. We also defined the time-domain operator $\hat{c}(t)$ as

$$\hat{c}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega,t_0)e^{i\omega t_0}e^{-i(\omega-\omega_0)t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega,0)e^{-i(\omega-\omega_0)t}. \tag{35}$$

and we have used the similar tricks as in deriving the master equation. Equation (34) is called the *quantum Heisenberg-Langevin equation*. Here note that $\hat{\alpha}(t)$ and $\hat{c}(t)$ have different dimensions, that is, [1] and $[\frac{1}{\sqrt{\text{time}}}]$.

1. *Relation to the master equation [3]*

The average value of Eq. (34) gives

$$\frac{d}{dt}\langle\hat{\alpha}(t)\rangle = -\left(i\Delta + \frac{\Gamma}{2}\right)\langle\hat{\alpha}(t)\rangle \tag{36}$$

since $\langle\hat{c}(t)\rangle = 0$ and the first term in Eq. (34) disappears. This equation can be obtained from the master equation Eq. (42) in *note 2016-12-05*, that is,

$$\frac{d}{dt}\langle\hat{a}(t)\rangle = \frac{d}{dt}\mathrm{Tr}\left\{\tilde{\sigma}(t)\hat{a}\right\} = \mathrm{Tr}\left\{\frac{d\tilde{\sigma}(t)}{dt}\hat{a}\right\} \tag{37}$$

with

$$\begin{aligned}\frac{\partial\tilde{\sigma}(t)}{\partial t} &= -\frac{i}{\hbar}\left[\hbar\Delta\hat{a}^\dagger\hat{a}, \tilde{\sigma}(t)\right] + \frac{\Gamma'+\Gamma}{2}\left(2\hat{a}\tilde{\sigma}(t)\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\tilde{\sigma}(t) - \sigma(t)\hat{a}^\dagger\hat{a}\right) \\ &\quad + \frac{\Gamma'}{2}\left(2\hat{a}^\dagger\tilde{\sigma}(t)\hat{a} - \hat{a}\hat{a}^\dagger\tilde{\sigma}(t) - \tilde{\sigma}(t)\hat{a}\hat{a}^\dagger\right).\end{aligned} \tag{38}$$

Using the invariance of the trace in a *circular* permutation and the commutator $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ we have indeed

$$\begin{aligned}\frac{d}{dt}\langle\hat{a}(t)\rangle &= \left\langle\frac{d\tilde{\sigma}(t)}{dt}\hat{a}\right\rangle \\ &= -i\Delta\langle\hat{a}\tilde{\sigma}\rangle - \frac{\Gamma}{2}\langle\hat{a}\tilde{\sigma}\rangle \\ &= -\left(i\Delta + \frac{\Gamma}{2}\right)\langle\hat{a}(t)\rangle.\end{aligned} \tag{39}$$

Note that the photon-number-dependent Γ' is absent since the terms contain Γ' are canceled out here. This exhibits the characteristics of harmonic oscillators with equi-spaced energy level struture. This also makes it clear that the Heisenberg's approach is more simpler and effective than Schrödinger's in analyzing damped harmonic oscillators.

C. The input-output theory [1]

Let us study the quantum Langevin equation a bit further. Let the system-environment Hamiltonian again be

$$H = H_s + H_b + H_i, \tag{40}$$

with

$$\begin{aligned}H_s &= H_s(\hat{a}, \hat{a}^\dagger, \hat{b}(\omega), \hat{b}^\dagger(\omega), \cdots) &(41)\\ H_b &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hbar\omega\hat{c}^\dagger(\omega)\hat{c}(\omega) &(42)\\ H_i &= -i\hbar\sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega) - \hat{a}\hat{c}^\dagger(\omega)\right), &(43)\end{aligned}$$

where the system Hamiltonian now contains the other field operators $\hat{b}(\omega), \hat{b}^\dagger(\omega), \cdots$ suggesting the existence of other decay channels.

We are interested in the effect of the environment mode specified by the operators $\hat{c}(\omega)$ and $\hat{c}^\dagger(\omega)$ on the system which interacts not only the concerned bath mode but also the other environment modes. The equation of motion for the envionment mode is the same as before:

$$\dot{\hat{c}}(\omega, t) = -i\omega\hat{c}(\omega, t) + \sqrt{\Gamma}\hat{a}(t), \tag{44}$$

while that for the system becomes

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}\left[H_s, \hat{a}(t)\right] - \sqrt{\Gamma}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}c(\omega, t). \tag{45}$$

We can find two formal solutions for Eq. (44); one of which is the one we have already encountered and we shall call it the *input mode*,

$$\hat{c}_{in}(\omega, t) = e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0) + \sqrt{\Gamma}\int_{t_0}^{t}d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau), \tag{46}$$

which is defined by referring to the *past time* t_0, the other is the *output mode*,

$$\hat{c}_{out}(\omega,t) = e^{-i\omega(t-t_1)}\hat{c}(\omega,t_1) - \sqrt{\Gamma}\int_t^{t_1} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau), \tag{47}$$

which is defined by referring to the *future time* t_1. Plugging those solutions in Eq. (45) we have two equations of motions for the system:

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[H_s+\Delta,\hat{a}(t)] - \frac{\Gamma}{2}\hat{a}(t) - \sqrt{\Gamma}\hat{c}_{in}(t)e^{-i\Omega t} \tag{48}$$

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[H_s+\Delta,\hat{a}(t)] + \frac{\Gamma}{2}\hat{a}(t) - \sqrt{\Gamma}\hat{c}_{out}(t)e^{-i\Omega t}, \tag{49}$$

where the time-domain operators $\hat{c}_{in}(t)$ and $\hat{c}_{out}(t)$ are defined by

$$\hat{c}_{in}(t)e^{-i\Omega t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t_0)}\hat{c}(\omega,t_0) \tag{50}$$

$$\hat{c}_{out}(t)e^{-i\Omega t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t_1)}\hat{c}(\omega,t_1), \tag{51}$$

respectively. By subtracting Eq. (49) from Eq. (48) we have

$$0 = -\Gamma\hat{a}(t) - \sqrt{\Gamma}\hat{c}_{in}(t)e^{-i\Omega t} + \sqrt{\Gamma}\hat{c}_{out}(t)e^{-i\Omega t} \tag{52}$$

which leads to the very useful *input-output relation* [1]:

$$\hat{c}_{out}(t) = \hat{c}_{in}(t) + \sqrt{\Gamma}\hat{\alpha}(t) \tag{53}$$

with $\hat{a}(t) = \hat{\alpha}(t)e^{-i\Omega t}$. It should be emphasized that the dimension of $\hat{\alpha}(t)$ and that of $\hat{c}_{in}(t)$ and $\hat{c}_{out}(t)$ are differnet. $\hat{\alpha}(t)$ is the annihilation operator of LC circuit, that is, (0+1)-dimensional Bosonic field, while $\hat{c}_{in}(t)$ and $\hat{c}_{out}(t)$ are the annihilation operators of 1D transmission line, that is, (1+1)-dimensional Bosonic field. At the boundary between the (0+1)-dimensional Bosonic field and the (1+1)-dimensional Bosonic field the special kind of care represented by Eq. (53) must be taken. The quantum Heisenberg-Langevin equation (34) and the input-output relation make up a set of the most useful equations in treating macroscopic quantum phenomena, which is applicable to many open quantum systems where a (0+1)-dimensional system coupled to a continuum (d+1)-dimensional environment, where "d" is the spatial dimension of the environment.

The input-output relation (53) can be compared with the more explicit *classical input-output relation* for the LC circuit with 1D transmission line:

$$V_{out}(t) = V_{in}(t) + Z_p I(t), \tag{54}$$

which can be rewritten as

$$\frac{V_{out}(t)}{\sqrt{Z_p}} = \frac{V_{in}(t)}{\sqrt{Z_p}} + \sqrt{Z_p}I(t), \tag{55}$$

or more suggestive form with the flux variable $\varphi(t) = L_0 I(t)$:

$$\underbrace{\frac{V_{out}(t)}{\sqrt{Z_p}}}_{\sim\sqrt{\hbar\Omega}\hat{c}_{out}(t)e^{-i\Omega t}} = \underbrace{\frac{V_{in}(t)}{\sqrt{Z_p}}}_{\sim\sqrt{\hbar\Omega}\hat{c}_{in}(t)e^{-i\Omega t}} + \underbrace{\sqrt{\frac{Z_p}{L_0}}\frac{\varphi(t)}{\sqrt{L_0}}}_{\sim\sqrt{\Gamma}\sqrt{\hbar\Omega}\hat{\alpha}(t)e^{-i\Omega t}}, \tag{56}$$

and thus reproducing the input-output relation (53);

$$\hat{c}_{out}(t) = \hat{c}_{in}(t) + \sqrt{\Gamma}\hat{\alpha}(t). \tag{57}$$

Here we used

$$V_{in}(t) = V^{\leftarrow}(x=0,t) = -i\int_0^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{-i\omega t} - h.c.\right) \sim -i\sqrt{\frac{\hbar\Omega Z_p}{2}}\left(\hat{c}_{in}(t)e^{-i\Omega t} - h.c.\right) \tag{58}$$

$$V_{out}(t) = V^{\rightarrow}(x=0,t) = -i\int_{-\infty}^{0}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{-i\omega t} - h.c.\right) \sim -i\sqrt{\frac{\hbar\Omega Z_p}{2}}\left(\hat{c}_{out}(t)e^{-i\Omega t} - h.c.\right) \tag{59}$$

$$\varphi(t) = -i\sqrt{\frac{\hbar L_0\Omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) = -i\sqrt{\frac{\hbar L_0\Omega}{2}}\left(\hat{\alpha}e^{-i\Omega t} - \hat{\alpha}^\dagger e^{i\Omega t}\right), \tag{60}$$

and chose the terms evolving as $e^{-i\Omega t}$.

III. PROBLEM

A. Quantum regression theorem [3, 4]

Using the Hermitian conjugate of Eq. (34) and showing

$$\langle \hat{c}^\dagger(t)\hat{\alpha}(t')\rangle = 0 \tag{61}$$

with consideration of relevent time scales, prove that the two-time average $\langle \hat{\alpha}(t)\hat{\alpha}(t')\rangle$ obeys the equation of motion

$$\frac{d}{dt}\langle \hat{\alpha}^\dagger(t)\hat{\alpha}(t')\rangle = -\left(-i\Delta + \frac{\Gamma}{2}\right)\langle \hat{\alpha}^\dagger(t)\hat{\alpha}(t')\rangle. \tag{62}$$

Equation (62) is in the same form as the equation of motion for the one-time average $\langle \hat{a}^\dagger(t)\rangle$, that is,

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\rangle = -\left(-i\Delta + \frac{\Gamma}{2}\right)\langle \hat{a}^\dagger(t)\rangle, \tag{63}$$

which is the Hermitian conjugate of Eq. (36). The fact that the time evolution of the two-time averages are obtained from the one-time averages is called the *quantum regression theorem.*

B. Relation between the classical Langevin equation and quantum Heisenberg-langevin equation

From Eq. (34) we have the equation of motion for $\hat{a}(t)$:

$$\frac{d}{dt}\hat{a}(t) = -\sqrt{\Gamma}\hat{c}(t)e^{-i\omega_0 t} - \left(i\omega_0 + \frac{\Gamma}{2}\right)\hat{a}(t), \tag{64}$$

where Δ is absorbed in ω_0. Then using the quantum regression theorem we shall have the following time evolution of the two-time average

$$\frac{d}{dt}\langle \hat{a}^\dagger(t)\hat{a}(0)\rangle = -\left(i\omega_0 + \frac{\Gamma}{2}\right)\langle \hat{a}^\dagger(t)\hat{a}(0)\rangle. \tag{65}$$

Solving this differential equation, plugging it into the definition of the photon number spectral density:

$$S_n(\Omega) = \int_{-\infty}^{\infty} dt \langle \hat{a}^\dagger(t)\hat{\alpha}(0)\rangle e^{i\Omega t}, \tag{66}$$

and multiplying the unit energy $\hbar\Omega$, show that the energy spectral density Eq. (26), which was obtained by the more explicit argument with the circuit equation, is reproduced.

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