Feynman path integral

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From the viewpoints of Schrödinger and Heisenberg, we have studied a hormonic oscillator coupled to an Ohmic environment. We shall now venture into the third viewpoint based on the *path integral*, invented by R. F. Feynman, to look at the quantum dissipative system. Here we shall learn the Feynman path integral method for treating simple cases, namely, a free particle and a particle in a well, i.e., a simple harmonic oscillator. This serves as a preparation for treating the *macroscopic quantum tunneling*, i.e., a particle-like macroscopic degree of freedom undergoing quantum tunnelings, by the Feynman path integral method.

I. BASICS OF PATH INTEGRAL

A. Basic idea [1]

It is said [1] that Feyman's path integral method is inspired by the mysterious remark in Dirac's book (page 128) [2], which states that

$$\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} dtL(q,\dot{q})\right] \text{cooresponds to } \langle q_f,t_f|q_i,t_i\rangle, \tag{1}$$

where $L(q,\dot{q})$ is the classical Lagrangian of a partical of mass m in a 1-dimensional potential $V(q)$,

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q), \tag{2}$$

and $\langle q_f,t_f|q_i,t_i\rangle$ is the quantum probability amplitude for the particle to go from a space-time point (q_i,t_i) to (q_f,t_f).

The exact correspondance, in the end Feynman found, can indeed be written by the space-time integral

$$\begin{aligned} \langle q_f,t_f|q_i,t_i\rangle &= \int_{q_i}^{q_f} Dq \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} dtL(q,\dot{q})\right] \\ &= \int_{q_i}^{q_f} Dq \exp\left[\frac{i}{\hbar}S[q]\right], \end{aligned} \tag{3}$$

where

$$\int_{q_i}^{q_f} Dq = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}}\int_{-\infty}^{\infty} dq_{N-1}\int_{-\infty}^{\infty} dq_{N-2}\cdots\int_{-\infty}^{\infty} dq_1. \tag{4}$$

is a infinite-dimensional path integral with $\{q_f, q_{N-1}, q_{N-2}, \cdots, q_1, q_i\}$ representing a single path (trajectory) of the particle in a coordinate space and $S[q]$ is the action. Let us see how this *Feynman path integral*, Eq. (3), is emerged.

B. Integral over paths through phase space [3]

The quantum probability amplitude for the particle $\langle q_f,t_f|q_i,t_i\rangle$ in Eq. (3) was written in the Heisenberg picture. This can be rewritten in the Schrödinger picture as

$$\langle q_f,t_f|q_i,t_i\rangle = \langle q_f|\exp\left[-\frac{i}{\hbar}H(t_f-t_i)\right]|q_i\rangle, \tag{5}$$

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where

$$H = \frac{1}{2m}p^2 + V(q) \tag{6}$$

is the Hamiltonian with p being the momentum conjugate of q. Chopping the time interval $t \equiv t_f - t_i$ into $N \gg 1$ steps lead to

$$e^{-\frac{i}{\hbar}Ht} = \left[e^{-\frac{i}{\hbar}H\Delta t}\right]^N, \tag{7}$$

where $\Delta t = \frac{t}{N}$. Now supposing that Δt is very short time interval (compared to the dominant time scale of the Hamiltonian dynamics) so that we can factorize $e^{-\frac{i}{\hbar}H\Delta t}$ in Eq. (7) into an easily diagonalized form, that is,

$$\begin{aligned} e^{-\frac{i}{\hbar}H\Delta t} &\cong \left(1 - i\frac{H}{\hbar}\Delta t\right) + \mathcal{O}(\Delta t^2) \\ &= \left(1 - \frac{i}{\hbar}\frac{p^2}{2m}\Delta t\right)\left(1 - \frac{i}{\hbar}V(q)\Delta t\right) + \mathcal{O}(\Delta t^2) \\ &\cong e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t} + \mathcal{O}(\Delta t^2). \end{aligned} \tag{8}$$

We thus have

$$\langle q_f, t_f | q_i, t_i\rangle = \langle q_f| \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{1} \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{2} \cdots \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{N} |q_i\rangle. \tag{9}$$

Here we introduce the *resolution of identity*,

$$1 = \int dq_k |q_k\rangle\langle q_k| \int dp_k |p_k\rangle\langle p_k|, \tag{10}$$

and insert N of them into Eq. (9) leading to

$$\begin{aligned} \langle q_f, t_f | q_i, t_i\rangle = \langle q_f| \int dq_N |q_N\rangle\langle q_N| \int dp_N |p_N\rangle\langle p_N| \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{1} \int dq_{N-1} |q_{N-1}\rangle\langle q_{N-1}| \\ \int dp_{N-1} |p_{N-1}\rangle\langle p_{N-1}| \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{2} \int dq_{N-2} |q_{N-2}\rangle\langle q_{N-2}| \\ \cdots \int dp_1 |p_1\rangle\langle p_1| \underbrace{e^{-\frac{i}{\hbar}\frac{p^2}{2m}\Delta t}e^{-\frac{i}{\hbar}V(q)\Delta t}}_{N} |q_i\rangle. \end{aligned} \tag{11}$$

We can simplifies Eq. (11) as

$$\begin{aligned} \langle q_f, t_f | q_i, t_i\rangle = \int \prod_{k=1}^{N-1} dq_k \int \prod_{k=1}^{N} dp_k \ \langle q_f | p_N\rangle e^{-\frac{i}{\hbar}\frac{p_N^2}{2m}\Delta t} e^{-\frac{i}{\hbar}V(q_{N-1})\Delta t} \langle p_N | q_{N-1}\rangle \\ \langle q_{N-1} | p_{N-1}\rangle e^{-\frac{i}{\hbar}\frac{p_{N-1}^2}{2m}\Delta t} e^{-\frac{i}{\hbar}V(q_{N-2})\Delta t} \langle p_{N-1} | q_{N-2}\rangle \\ \cdots \ \langle q_1 | p_1\rangle e^{-\frac{i}{\hbar}\frac{p_1^2}{2m}\Delta t} e^{-\frac{i}{\hbar}V(q_i)\Delta t} \langle p_1 | q_i\rangle. \end{aligned} \tag{12}$$

Remembering that within the position representation

$$\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{qp}{\hbar}}, \tag{13}$$

Eq. (12) can be further simplified and given as a (2N-1)-dimensional integral

$$\begin{aligned} \langle q_f, t_f | q_i, t_i\rangle &= \int \prod_{k=1}^{N-1} dq_k \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} e^{-\frac{i}{\hbar}\left(\frac{p_N^2}{2m} + V(q_{N-1}) - p_N \frac{q_f - q_{N-1}}{\Delta t}\right)\Delta t} \\ &\quad e^{-\frac{i}{\hbar}\left(\frac{p_{N-1}^2}{2m} + V(q_{N-2}) - p_{N-1}\frac{q_{N-1} - q_{N-2}}{\Delta t}\right)\Delta t} \cdots e^{-\frac{i}{\hbar}\left(\frac{p_1^2}{2m} - V(q_i) + p_1 \frac{q_1 - q_i}{\Delta t}\right)\Delta t} \\ &= \int \prod_{k=1}^{N-1} dq_k \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \exp\left[-\frac{i}{\hbar}\Delta t \sum_{k=0}^{N-1}\left(\frac{p_{k+1}^2}{2m} + V(q_k) - p_{k+1}\frac{q_{k+1} - q_k}{\Delta t}\right)\right], \end{aligned} \tag{14}$$

where we set $q_0 = q_i$.

Now by taking the continuum limit, that is, $N \to \infty$ while keeping $t = N\Delta t$ constant, we have

$$
\begin{aligned}
\langle q_f, t_f | q_i, t_i \rangle &= \underbrace{\int \prod_{k=1}^{N-1} dq_k}_{\int Dq} \underbrace{\int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}}_{\int Dp} \exp\left[-\frac{i}{\hbar} \int_0^t dt' \left(\frac{p(t')^2}{2m} + V(q(t')) - p(t')\dot{q}(t') \right) \right] \\
&= \int Dq \int Dp \;\exp\left[\frac{i}{\hbar} \int_0^t dt' \left(p(t')\dot{q}(t') - H(q(t'), p(t')) \right) \right], \qquad (15)
\end{aligned}
$$

where we used

$$
\Delta t \sum_{k=0}^{N-1} \Rightarrow \int_0^t dt' \qquad (16)
$$

$$
\frac{q_{k+1} - q_k}{\Delta t} \Rightarrow \dot{q}(t')|_{t'=k\Delta t}, \qquad (17)
$$

with $\Rightarrow$ indicating the continuum limit. Equation (15) is the *Hamiltonian formulation of the path integral.*

C. Integral over paths through coordinate space [3]

The Hamiltonian formulation of the path integral, Eq. (15) represents Feynman's idea that the quantum probability amplitude $\langle q_f, t_f | q_i, t_i \rangle$ can be obtained by *summing over all possible paths in the phase space.* There is an analogous formula based on Lagrangian and the philosophy is to get $\langle q_f, t_f | q_i, t_i \rangle$ by *summing over all possible paths in the configuration space.* To this end, we just need to carry out the integration over Dp in Eq. (15). This can be done by the following procedure. First, rewrite the path integral as

$$
\langle q_f, t_f | q_i, t_i \rangle = \int Dq \exp\left[-\frac{i}{\hbar} \int_0^t dt' V(q) \right] \int Dp \;\exp\left[-\frac{i}{\hbar} \int_0^t dt' \left(\frac{p^2}{2m} - p\dot{q} \right) \right], \qquad (18)
$$

and recognize that the second integrand is quadratic in p. Second, to execute the integration over p with *Gaussian integration* (see Appendix) go back to the finite-dimensional integral form,

$$
\begin{aligned}
\int Dp \;\exp\left[-\frac{i}{\hbar} \int_0^t dt' \left(\frac{p^2}{2m} - p\dot{q} \right) \right] &\Rightarrow \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \exp\left[-\frac{i}{\hbar} \Delta t \sum_{k=1}^{N} \left(\frac{p_k^2}{2m} - p_k \dot{q}_k \right) \right] \\
&= \left(\frac{1}{2\pi\hbar} \right)^N \int d\boldsymbol{p} \exp\left[-\frac{1}{2} \left(\boldsymbol{p}^T \mathbf{A} \boldsymbol{p} \right) + \boldsymbol{j}^T \boldsymbol{p} \right], \qquad (19)
\end{aligned}
$$

where

$$\boldsymbol{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_N \end{bmatrix}; \tag{20}$$

$$\mathbf{A} = \begin{bmatrix} \frac{i}{m\hbar}\Delta t & 0 & 0 & \cdots & 0 \\ 0 & \frac{i}{m\hbar}\Delta t & 0 & \cdots & 0 \\ 0 & 0 & \frac{i}{m\hbar}\Delta t & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \frac{i}{m\hbar}\Delta t \end{bmatrix}; \tag{21}$$

$$\boldsymbol{j} = \begin{bmatrix} \frac{i}{\hbar}\dot{q}_1\Delta t \\ \frac{i}{\hbar}\dot{q}_2\Delta t \\ \frac{i}{\hbar}\dot{q}_3\Delta t \\ \vdots \\ \frac{i}{\hbar}\dot{q}_N\Delta t \end{bmatrix}. \tag{22}$$

Third, perform the Gaussian integration (see Eq. (A4)):

$$\left(\frac{1}{2\pi\hbar}\right)^N \int d\boldsymbol{p} \exp\left[-\frac{1}{2}\left(\boldsymbol{p}^T\mathbf{A}\boldsymbol{p}\right) + \boldsymbol{j}^T\boldsymbol{p}\right] = \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}} \exp\left[-\frac{i}{\hbar}\Delta t\sum_{k=1}^{N}\left(-\frac{1}{2}m\dot{q}_k^{\,2}\right)\right]. \tag{23}$$

Finally, by taking the continuum limit again we can complete the integration over Dp as

$$\int Dp\ \exp\left[-\frac{i}{\hbar}\int_0^t dt'\left(\frac{p^2}{2m} - p\dot{q}\right)\right] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}} \exp\left[-\frac{i}{\hbar}\int_0^t dt'\left(-\frac{1}{2}m\dot{q}_k^{\,2}\right)\right]. \tag{24}$$

By plugging Eq. (24) into Eq. (18) we rearch the same conclusion as Feynman, i.e., Eq. (3)!

D. Example: free particle

Having get the beautiful formula Eq. (3), this formula per se is little use. Consider the the simplest example, free particle with mass m. In this case the Hamiltonian is

$$H = \frac{p^2}{2m}. \tag{25}$$

We shall now see that even in this simplest case the calculation of $G_{\text{free}}(q_f, q_i; t) \equiv \langle q_f, t_f | q_i, t_i\rangle$ with the Feynman path integral method is rather clumsy and cumbersome. We shall see the true power of the Feynman path integral method later on.

To avoid the divergence problem inherent in the path integral in the continuum limit [3], the starting point to get the formula of $G_{\text{free}}(q_f, q_i; t)$ is again the discretized finite-dimensional integral, Eq (14):

$$\int \prod_{k=1}^{N-1} dq_k \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\sum_{k=1}^{N}\left(p_k\left(q_k - q_{k-1}\right) - \frac{p_k^2}{2m}\Delta t\right)\right]. \tag{26}$$

Here we notice that the integrations over $\{q_1, q_2, \cdots, q_{N-1}\}$ are separately perfermed and

$$\int dq_k \exp\left[\frac{i}{\hbar} q_k \left(p_k - p_{k+1}\right)\right] = 2\pi\hbar\delta_{p_k p_{k+1}}. \tag{27}$$

Thus Eq. (26) becomes

$$(2\pi\hbar)^{N-1} \int \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\, \delta_{p_k p_{k+1}} \exp\left[\frac{i}{\hbar}\left(\underbrace{(p_N q_N - p_1 q_0)}_{\text{leftover}} + \sum_{k=1}^{N}\left(-\frac{p_k^2}{2m}\Delta t\right)\right)\right]. \tag{28}$$

Performing the integration over p_N we have

$$(2\pi\hbar)^{N-2} \int \prod_{k=1}^{N-1} \frac{dp_k}{2\pi\hbar}\, \delta_{p_k p_{k+1}} \exp\left[\frac{i}{\hbar}\left((p_{N-1} q_N - p_1 q_0) + \sum_{k=1}^{N-1}\left(-\frac{p_k^2}{2m}\Delta t\right) - \frac{p_{N-1}^2}{2m}\Delta t\right)\right]. \tag{29}$$

Then performing the integration over p_{N-1} we have

$$(2\pi\hbar)^{N-3} \int \prod_{k=1}^{N-2} \frac{dp_k}{2\pi\hbar}\, \delta_{p_k p_{k+1}} \exp\left[\frac{i}{\hbar}\left((p_{N-2} q_N - p_1 q_0) + \sum_{k=1}^{N-2}\left(-\frac{p_k^2}{2m}\Delta t\right) - \frac{p_{N-2}^2}{2m}2\Delta t\right)\right]. \tag{30}$$

Iteration of this integration over p_k up to $k = 2$ leads to

$$\begin{aligned} G_{\text{free}}(q_f, q_i; t) &= \lim_{N\to\infty} \int \frac{dp_1}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left((p_1 q_N - p_1 q_0) + \left(-\frac{p_1^2}{2m}\Delta t\right) - \frac{p_1^2}{2m}(N-1)\Delta t\right)\right] \\ &= \lim_{N\to\infty} \int \frac{dp_1}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left((q_N - q_0)\, p_1 - \frac{t}{2m} p_1^2\right)\right] \\ &= \int \frac{dp_1}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left((q_f - q_i)\, p_1 - \frac{t}{2m} p_1^2\right)\right], \end{aligned} \tag{31}$$

where $q_0 = q_i$ and $q_N = q_f$. This is the Gaussian-form integral with respect to p_1. Performing the Gaussian integration over p_1 (see Eq. (A2)) we have

$$G_{\text{free}}(q_f, q_i; t) = \sqrt{\frac{1}{4\pi\left(\frac{i\hbar}{2m}\right)t}} \exp\left[-\frac{(q_f - q_i)^2}{4\left(\frac{i\hbar}{2m}\right)t}\right]\Theta(t), \tag{32}$$

where the step function, $\Theta(t)$, is introduced to account for the causality. Note that this is like a solution of a classical diffusion equation with the diffusion constant $D = \frac{i\hbar}{2m}$.

II. STATIONARY PHASE APPROXIMATION TO THE PATH INTEGRAL [3]

The true power of the Feynman path integral method can be seen when the *semi-classical limits* of quantum theories are dealt with. This includes the situation where a macroscopic objest is rest at a classical equilibrium position and the quantum fluctuations around it are asked.

To see how the solutions of classical equations of motion appear in the path integral, let us explore the *stationary phase (saddle-point) approximation* to the path integral. The first step is to find the solutions of the classical equation of motion associated with the Lagrangian $L(q, \dot{q})$, that is, the Euler-Lagrange equation, $\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial L(q,\dot{q})}{\partial q} = 0$, that is, for $L(q, \dot{q})$ in Eq. (2)

$$m\ddot{q} + \frac{\partial V(q)}{\partial q} = 0. \tag{33}$$

As the second step, let q_{cl} be a only solution of Eq. (33) and set $q = q_{cl} + r$. The action $S[q] \equiv \int_0^t dt' L(q, \dot{q})$ in

Eq. (3) can then be Taylor-expanded as

$$\begin{aligned}
S[q] &= \int_0^t dt' L(q,\dot{q}) \\
&= S[q_{cl}] + \int_0^t dt' \underbrace{\frac{\delta S[q_{cl}]}{\delta q(t')}}_{0} r(t') + \frac{1}{2}\int_0^t dt' \int_0^t dt'' r(t') \frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')} r(t'') + \cdots \\
&\simeq S[q_{cl}] + \frac{1}{2}\int_0^t dt' \int_0^t dt'' r(t') \frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')} r(t''), \qquad (34)
\end{aligned}$$

where $\frac{\delta S[q_{cl}]}{\delta q(t')} = 0$ is ensured by the classical solution q_{cl}. Here $\frac{\delta S[q_{cl}]}{\delta q(t')}$ and $\frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')}$ are the *functional derivatives*. The meaning of the functional derivatives can be made clear by (again) going back to the discrete N-dimensional version of Eq. (34)

$$\begin{aligned}
S[q] &= \Delta t \sum_{k=1}^{N} L(q_k, \dot{q}_k) \\
&= \Delta t \sum_{k=1}^{N} L(q_k^{(cl)}, \dot{q}_k{}^{(cl)}) + \Delta t \sum_{l=1}^{N} \frac{\partial \left(\Delta t \sum_k L(q_k^{(cl)}, \dot{q}_k{}^{(cl)})\right)}{\partial q_l} r_l + \frac{1}{2}\Delta t \sum_{l=1}^{N} \Delta t \sum_{m=1}^{N} \frac{\partial^2 \left(\Delta t \sum_k L(q_k^{(cl)}, \dot{q}_k{}^{(cl)})\right)}{\partial q_l \partial q_m} r_l r_m \\
&= S[q^{(cl)}] + \Delta t \sum_{l=1}^{N} \frac{\partial S[q^{(cl)}]}{\partial q_l} r_l + \frac{1}{2}\Delta t \sum_{l=1}^{N} \Delta t \sum_{m=1}^{N} \frac{\partial^2 S[q^{(cl)}]}{\partial q_l \partial q_m} r_l r_m \qquad (35)
\end{aligned}$$

and by taking the continuum limit

$$\Delta t \sum_{l=1}^{N} \frac{\partial S[q^{(cl)}]}{\partial q_l} r_l \Rightarrow \int_0^t dt' \frac{\delta S(q(t'))}{\delta q(t')} r(t') \qquad (36)$$

$$\frac{1}{2}\Delta t \sum_{l=1}^{N} \Delta t \sum_{m=1}^{N} \frac{\partial^2 S[q^{(cl)}]}{\partial q_l \partial q_m} r_l r_m \Rightarrow \frac{1}{2}\int_0^t dt' \int_0^t dt'' r(t') \frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')} r(t''). \qquad (37)$$

Finally, by plugging Eq. (34) into Eq. (3) we have the *stationary phase (saddle-point)* approximation to the path integral:

$$\begin{aligned}
\langle q_f, t_f | q_i, t_i \rangle &= \int_{q_i}^{q_f} Dq \exp\left[\frac{i}{\hbar} S[q]\right] \\
&= \underbrace{\exp\left[\frac{i}{\hbar} S[q_{cl}]\right]}_{\text{classical path}} \underbrace{\int_{q_i}^{q_f} Dr \exp\left[\frac{1}{2}\int_0^t dt' \int_0^t dt'' r(t') \frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')} r(t'')\right]}_{\text{quantum fluctuation}}. \qquad (38)
\end{aligned}$$

The form Eq. (38) appears to be very appealing: the classical path associated with the classical action $S[q_{cl}]$ are embellished with the quantum fluctuation. Note that the quantum fluctuation is now completely described by c-numbers as opposed to quantum operators.

To make further progress let us see more explicit from with $L(q,\dot{q})$ in Eq. (2). By expanding the action $S[q_{cl}]$ in

$r(t)$ explicitly we have

$$\begin{aligned}
S[q] &= \int_0^\infty dt' \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) \\
&\simeq \int_0^t dt' \left[\frac{1}{2}m\left(\dot{q_{cl}}^2 + 2\dot{q_{cl}}\dot{r} + \dot{r}^2\right) - \left(V(q_{cl}) + \frac{\partial V(q_{cl})}{\partial q}r + \frac{1}{2}\frac{\partial^2 V(q_{cl})}{\partial q^2}r^2\right)\right] \\
&= \int_0^t dt' \left[\frac{1}{2}m\dot{q_{cl}}^2 - V(q_{cl})\right] + \int_0^t dt' \left[m\dot{q_{cl}}\dot{r} - \frac{\partial V(q_{cl})}{\partial q}r\right] + \int_0^t \left[\frac{1}{2}m\dot{r}^2 - \frac{1}{2}\frac{\partial^2 V(q_{cl})}{\partial q^2}r^2\right] \\
&= S[q_{cl}] - \int_0^t dt' \underbrace{\left[m\ddot{q_{cl}} + \frac{\partial V(q_{cl})}{\partial q}\right]}_{0} r(t') - \frac{1}{2}\int_0^t dt' r(t') \left[m\frac{d^2}{dt'^2} + \frac{\partial^2 V(q_{cl})}{\partial q^2}\right] r(t') \\
&= S[q_{cl}] - \frac{1}{2}\int_0^t dt' r(t') \left[m\frac{d^2}{dt'^2} + \frac{\partial^2 V(q_{cl})}{\partial q^2}\right] r(t'),
\end{aligned} \tag{39}$$

where, in the third line, we performed the integrations by part,

$$\int_0^t dt' m\dot{q_{cl}}\dot{r} = \underbrace{[m\dot{q_{cl}}r]_0^t}_{0} - \int_0^t dt' m\ddot{q_{cl}}r \tag{40}$$

$$\int_0^t dt' m\dot{r}^2 = \underbrace{[m\dot{r}r]_0^t}_{0} - \int_0^t dt' m\ddot{r}r. \tag{41}$$

By compared with Eq. (34) we obtain the following relation:

$$\frac{1}{2}\int_0^t dt' \int_0^t dt'' r(t') \frac{\delta^2 S[q_{cl}]}{\delta q(t')\delta q(t'')} r(t'') = -\frac{1}{2}\int_0^t dt' r(t') \left[m\frac{d^2}{dt'^2} + \frac{\partial^2 V(q_{cl})}{\partial q^2}\right] r(t'). \tag{42}$$

A. Example: quantum harmonic oscillator

Let us apply the above argument to a particle in a harmonic potential $V = \frac{1}{2}kq^2$, that is, a harmonic oscillator. The classical equation of motion is $m\ddot{q} + kq = 0$. Imposing the boundary conditions $q(0) = q(t) = 0$, the solution of the classical motion is obiously $q_{cl} = 0$. We thus have

$$\begin{aligned}
G_{\mathrm{HO}}(0,0;t) \equiv \langle q_f = 0, t | q_i = 0, 0\rangle &= \int Dq \exp\left[\frac{i}{\hbar}S[q]\right] \\
&\simeq \underbrace{\exp\left[\frac{i}{\hbar}S[q_{cl}]\right]}_{1} \int Dr \exp\left[-\frac{i}{\hbar}\int_0^t dt' r(t') \left[\frac{1}{2}\left(m\frac{d^2}{dt'^2} + \frac{\partial^2 V(q_{cl})}{\partial q^2}\right)\right] r(t')\right] \\
&= \int Dr \exp\left[-\frac{i}{\hbar}\int_0^t dt' r(t') \frac{m}{2}\left(\frac{d^2}{dt'^2} + \omega^2\right) r(t')\right],
\end{aligned} \tag{43}$$

where $\omega = \sqrt{\frac{k}{m}}$ is the eigenfrequency of the oscillator. This integral is again Gaussian form, so we can perform the Gaussian integration. To perform the integral let us tentatively assume the differential operator $-\frac{m}{2}\left(\frac{d^2}{dt'^2} + \omega^2\right)$ be a finite-dimensional matrix $\mathbf{A}$. The integral then becomes familiar one as Eq. (A3) and get

$$G_{\mathrm{HO}}(0,0;t) = \mathcal{N}\frac{1}{\sqrt{\det[\mathbf{A}]}}, \tag{44}$$

with $\mathcal{N}$ absorbed several constants, which may be divergent after taking the continuum limit, though. Then the question is; what is $\det[\mathbf{A}]$? The answer can be found by expressing $\mathbf{A}$ in terms of eigenvalues, that is,

$$\begin{aligned}
\mathbf{A}v_n &\equiv -\frac{m}{2}\left(\frac{d^2}{dt'^2} + \omega^2\right) v_n \\
&= \epsilon_n v_n.
\end{aligned} \tag{45}$$

The eigestates v_n are given by

$$v_n = \sin\left(\frac{n\pi t'}{t}\right) \tag{46}$$

with the eigenvalues

$$\epsilon_n = \frac{m}{2}\left(-\omega^2 + \left(\frac{n\pi}{t}\right)^2\right) \tag{47}$$

for $n = 1, 2, \cdots \infty$. Thus the determinant of $\mathbf{A}$ is thus given by

$$\det[\mathbf{A}] = \prod_{n=1}^{\infty} \epsilon_n = \prod_{n=1}^{\infty} \frac{m}{2}\left(-\omega^2 + \left(\frac{n\pi}{t}\right)^2\right). \tag{48}$$

We then notice that $\frac{1}{\sqrt{\det[\mathbf{A}]}}$ is obtained from the infinite product of $\left(-\omega^2 + \left(\frac{n\pi}{t}\right)^2\right)^{-\frac{1}{2}}$, each of which is divergent for $\frac{n\pi}{t} = \omega$, a very alarming situation!

To circumvent the calculation of the dangerous determinant explicitly, we can exploit the well-behaved result obtained for a free particle in Sec. I D. Indeed, $G_{\text{free}}(0,0;t)$ is the special case of $G_{\text{HO}}(0,0;t)$ for $V(q) = 0$, that is, $\omega = 0$. Let us evaluate the following quantity,

$$G_{\text{HO}}(0,0;t) = \left(\frac{G_{\text{HO}}(0,0;t)}{G_{\text{free}}(0,0;t)}\right) G_{\text{free}}(0,0;t). \tag{49}$$

The quantity inside the parentheses in Eq. (49) gives

$$\begin{aligned} \frac{G_{\text{HO}}(0,0;t)}{G_{\text{free}}(0,0;t)} &= \frac{\mathcal{N}\prod_{n=1}^{\infty}\left[\frac{m}{2}\left(-\omega^2 + \left(\frac{n\pi}{t}\right)^2\right)\right]^{-\frac{1}{2}}}{\mathcal{N}\prod_{n=1}^{\infty}\left[\frac{m}{2}\left(\frac{n\pi}{t}\right)^2\right]^{-\frac{1}{2}}} \\ &= \prod_{n=1}^{\infty}\left[1 - \left(\frac{\omega t}{n\pi}\right)^2\right]^{-\frac{1}{2}} = \sqrt{\frac{\omega t}{\sin(\omega t)}}. \end{aligned} \tag{50}$$

Thus, with Eq.(49), $G_{\text{HO}}(0,0;t)$ bocomes

$$G_{\text{HO}}(0,0;t) = \sqrt{\frac{\omega t}{\sin(\omega t)}} G_{\text{free}}(0,0;t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}}\Theta(t), \tag{51}$$

where we used Eq. (32) for $G_{\text{free}}(0,0;t)$.

III. PROBLEM

A. Example: free particle in momentum representation

Consider again the path integral of a free particle with mass m with $H = \frac{p^2}{2m}$. This time, however, we are interested in a form in the momentum representation, that is,

$$\langle p', t|p, 0\rangle = \int dq \int dq' \langle q', t|q, 0\rangle \exp\left[-\frac{i}{\hbar}(pq + p'q')\right]. \tag{52}$$

Show that

$$\langle p', t|p, 0\rangle = 2\pi\hbar\delta(p' - p)\exp\left[-\frac{i}{\hbar}\left(\frac{p^2}{2m}\right)t\right]. \tag{53}$$

Appendix A: Gaussian integration

First, some mathematics. The most fundamental Gaussian integration is

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}. \tag{A1}$$

An interesting and useful Gaussian integration is

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}. \tag{A2}$$

The Multi-dimensional expansion of Eq. (A1) is

$$\int_{-\infty}^{\infty} d\boldsymbol{v} e^{-\frac{1}{2}\boldsymbol{v}^T \mathbf{A} \boldsymbol{v}} = (2\pi)^{\frac{N}{2}} \frac{1}{\sqrt{\det[\mathbf{A}]}}, \tag{A3}$$

and that of Eq. (A2) is

$$\int_{-\infty}^{\infty} d\boldsymbol{v} e^{-\frac{1}{2}\boldsymbol{v}^T \mathbf{A} \boldsymbol{v} + \boldsymbol{j}\cdot\boldsymbol{v}} = (2\pi)^{\frac{N}{2}} \frac{1}{\sqrt{\det[\mathbf{A}]}} e^{\frac{1}{2}\boldsymbol{j}^T \mathbf{A}^{-1} \boldsymbol{j}}, \tag{A4}$$

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