Master equation

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We have learned how to treat the dissipative element, that is, the *environment*, quantum mechanically. Now we turn our attention to the *system* and learn how the dynamics of the system is affected by the environment. We shall start by considering, from the viewpoint of Schrödinger, a hormonic oscillator as the system, which is coupled to an Ohmic environment, and analyze the resultant master equation for the damped harmonic oscillator.

I. "COARSE-GRAINED" TIME EVOLUTION [1]

Let us reexamine the LCR circuit from the viewpoint of Hamiltonian formalism. We shall assume the total Hamiltonian to be

$$H = H_s + H_b + H_I, \tag{1}$$

where H_s, H_b, and H_I are the Hamiltonians of the LC circuit (the system), the transmission line (the Ohmic environment), and their interaction, and are respectively given by

$$H_s = \hbar\omega_0 \hat{a}^\dagger \hat{a} \tag{2}$$

$$H_b = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hbar\omega \hat{c}^\dagger(\omega)\hat{c}(\omega) \tag{3}$$

$$\begin{aligned} H_I &= -i\hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(f(\omega)\hat{a}^\dagger \hat{c}(\omega) - f^*(\omega)\hat{a}\hat{c}^\dagger(\omega) \right) \\ &= -i\hbar \left[\hat{a}^\dagger \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f(\omega)\hat{c}(\omega)}_{R^-} - \hat{a} \underbrace{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f^*(\omega)\hat{c}^\dagger(\omega)}_{R^+} \right] \\ &= -i\hbar \left[\hat{a}^\dagger R^- - \hat{a} R^+ \right]. \end{aligned} \tag{4}$$

Here the coupling strength $f(\omega)$ has the dimension of $\sqrt{\text{Angular frequency}}$ and will be identified to the familiar quantity later on.

Let us analyze the coupled system by focusing on the evolution of the density operator ρ of global system. The time evolution equation in the Schrödinger representation is

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\left[H, \rho(t)\right]. \tag{5}$$

In the interaction representation with respenct to $H_s + H_b$, it becomes

$$\frac{d}{dt}\tilde{\rho(t)} = \frac{1}{i\hbar}\left[\tilde{H}_I(t), \tilde{\rho(t)}\right], \tag{6}$$

where

$$\tilde{\rho}(t) = e^{i\frac{H_s + H_b}{\hbar}t} \rho(t) e^{-i\frac{H_s + H_b}{\hbar}t} \tag{7}$$

$$\begin{aligned} \tilde{H}_I(t) &= e^{i\frac{H_s + H_b}{\hbar}t} H_I e^{-i\frac{H_s + H_b}{\hbar}t} \\ &= -i\hbar \left[\hat{a}^\dagger e^{i\omega_0 t} \tilde{R}^-(t) - \hat{a} e^{-i\omega_0 t} \tilde{R}^+(t) \right], \end{aligned} \tag{8}$$

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where

$$\tilde{R}^-(t) = e^{i\frac{H_s+H_b}{\hbar}t}R_- e^{-i\frac{H_s+H_b}{\hbar}t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f(\omega)\hat{c}(\omega)e^{-i\omega t} \tag{9}$$

$$\tilde{R}^+(t) = e^{i\frac{H_s+H_b}{\hbar}t}R_+ e^{-i\frac{H_s+H_b}{\hbar}t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f^*(\omega)\hat{c}^\dagger(\omega)e^{i\omega t}. \tag{10}$$

The benefit of turning to the interaction representation is that if the interaction H_I is sufficiently small $\tilde{\rho}(t)$ evolves slowly.

By formally integrating Eq. (6) we have

$$\tilde{\rho}(t+\Delta t) = \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{H}_I(t'), \tilde{\rho}(t')\right] \tag{11}$$

which leads to the following iterative relations:

$$\begin{aligned}\Delta\tilde{\rho}(t) &\equiv \tilde{\rho}(t+\Delta t) - \tilde{\rho}(t)\\ &= \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{H}_I(t'), \tilde{\rho}(t')\right]\\ &= \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{H}_I(t'), \tilde{\rho}(t)\right] + \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t} dt'\int_t^{t'} dt'' \left[\tilde{H}_I(t'), \left[\tilde{H}_I(t''), \tilde{\rho}(t'')\right]\right]\\ &= \cdots .\end{aligned} \tag{12}$$

Let us now assume that the interaction between the system and the environment is small and retain the terms in Eq. (12) up to second order in $\tilde{H}_I$:

$$\Delta\tilde{\rho}(t) = \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{H}_I(t'), \tilde{\rho}(t)\right] + \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t} dt'\int_t^{t'} dt'' \left[\tilde{H}_I(t'), \left[\tilde{H}_I(t''), \tilde{\rho}(t)\right]\right]. \tag{13}$$

Now we make a crucial assumption. By writting the reduced density operators $\tilde{\sigma}(t) = \mathrm{Tr}_b\{\rho(t)\}$ and $\tilde{\sigma}_b(t) = \mathrm{Tr}_s\{\rho(t)\}$ for the system (LC) and environment (R), respectively, let us assume

$$\tilde{\rho}(t) = \tilde{\sigma}(t)\otimes\tilde{\sigma}_b(t), \tag{14}$$

meaning that the correlations between the system (LC) and the environment (R) at time t are neglected as they disappear very quickly compared with the typical time scale Δt we are interested in. Further assume that the environment is not affected by the coupling to the system and thus the environment is in a stationary state, that is,

$$\tilde{\sigma}_b(t) = \tilde{\sigma}_b(0) = \sigma_b \tag{15}$$

and thus

$$[\sigma_b, H_b] = 0. \tag{16}$$

Now, by tracing out the environmental degrees of freedom we have

$$\mathrm{Tr}_b\{\Delta\tilde{\rho}(t)\} = \left(\frac{1}{i\hbar}\right)^2\int_t^{t+\Delta t} dt'\int_t^{t'} dt''\,\mathrm{Tr}_b\left\{\left[\tilde{H}_I(t'), \left[\tilde{H}_I(t''), \tilde{\rho}(t)\right]\right]\right\}, \tag{17}$$

where

$$\mathrm{Tr}_b\{\cdots\} = \sum_n \langle n|\cdots|n\rangle \tag{18}$$

with $\{|n\rangle\}$ being the complete set of states for the environment. Here the first term in Eq. (13) disappears since

$$\begin{aligned}\mathrm{Tr}_b\left\{\left[\tilde{H}_I(t'), \tilde{\rho}(t)\right]\right\} &= \mathrm{Tr}_b\left\{\left[\tilde{H}_I(t'), \tilde{\sigma}(t)\otimes\sigma_b\right]\right\}\\ &= -i\hbar\left[\hat{a}^\dagger e^{i\omega_0 t'}\mathrm{Tr}_b\left\{\tilde{R}^-(t')\sigma_b\right\} - \hat{a}e^{-i\omega_0 t'}\mathrm{Tr}_b\left\{\tilde{R}^+(t')\sigma_b\right\}\right]\tilde{\sigma}(t)\\ &\quad -i\hbar\tilde{\sigma}(t)\left[\hat{a}^\dagger e^{i\omega_0 t'}\mathrm{Tr}_b\left\{\tilde{R}^-(t')\sigma_b\right\} - \hat{a}e^{-i\omega_0 t'}\mathrm{Tr}_b\left\{\tilde{R}^+(t')\sigma_b\right\}\right].\end{aligned} \tag{19}$$

In case where the environment is in the thermal equilibrium

$$\mathrm{Tr}_b\left\{\tilde{R}^+(t')\sigma_b\right\} = \mathrm{Tr}_b\left\{\tilde{R}^-(t')\sigma_b\right\} = 0, \tag{20}$$

thus

$$\mathrm{Tr}_b\left\{\left[\tilde{H}_I(t'), \tilde{\rho}(t)\right]\right\} = 0. \tag{21}$$

The *coarse-grained* time evolution of $\tilde{\sigma}(t)$ can then be given by

$$\begin{aligned}
\frac{\Delta\tilde{\sigma}(t)}{\Delta t} &\equiv \frac{\mathrm{Tr}_b\left\{\Delta\tilde{\rho}(t)\right\}}{\Delta t} \\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \mathrm{Tr}_b\left\{\left[\tilde{H}_I(t'), \left[\tilde{H}_I(t''), \tilde{\sigma}(t)\otimes\sigma_b\right]\right]\right\} \\
&= -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \left[\mathrm{Tr}_b\left\{\tilde{H}_I(t')\tilde{H}_I(t'')\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\right\}\right. \\
&\qquad -\mathrm{Tr}_b\left\{\tilde{H}_I(t')\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\tilde{H}_I(t'')\right\} \\
&\qquad -\mathrm{Tr}_b\left\{\tilde{H}_I(t'')\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\tilde{H}_I(t')\right\} \\
&\qquad \left.+\mathrm{Tr}_b\left\{\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\tilde{H}_I(t'')\tilde{H}_I(t')\right\}\right].
\end{aligned} \tag{22}$$

II. LINDBLAD FORM

A. Master equation [1–3]

Let us further investigate the coarse-grained time evolution of the reduced density operator $\sigma(t)$ in Eq. (22) to obtain the so-called *Lindblad form*, the non-unitary time evolution which nonetheless preserves complete positivity and trance of $\sigma(t)$. By plugging Eq. (8) into Eq. (22) we get

$$\begin{aligned}
\frac{\Delta\tilde{\sigma}(t)}{\Delta t} &= \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \,[\\
&\quad \mathrm{Tr}_b\left\{\left(\tilde{a}^\dagger(t')\tilde{R}^-(t') - \tilde{a}(t')\tilde{R}^+(t')\right)\left(\tilde{a}^\dagger(t'')\tilde{R}^-(t'') - \tilde{a}(t'')\tilde{R}^+(t'')\right)\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\right\} \\
&\quad -\mathrm{Tr}_b\left\{\left(\tilde{a}^\dagger(t')\tilde{R}^-(t') - \tilde{a}(t')\tilde{R}^+(t')\right)\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\left(\tilde{a}^\dagger(t'')\tilde{R}^-(t'') - \tilde{a}(t'')\tilde{R}^+(t'')\right)\right\} \\
&\quad -\mathrm{Tr}_b\left\{\left(\tilde{a}^\dagger(t'')\tilde{R}^-(t'') - \tilde{a}(t'')\tilde{R}^+(t'')\right)\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\left(\tilde{a}^\dagger(t')\tilde{R}^-(t') - \tilde{a}(t')\tilde{R}^+(t')\right)\right\} \\
&\quad +\mathrm{Tr}_b\left\{\left(\tilde{\sigma}(t)\otimes\sigma_b\right)\left(\tilde{a}^\dagger(t'')\tilde{R}^-(t'') - \tilde{a}(t'')\tilde{R}^+(t'')\right)\left(\tilde{a}^\dagger(t')\tilde{R}^-(t') - \tilde{a}(t')\tilde{R}^+(t')\right)\right\} \\
&\quad] \\
&= \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \,[\\
&\quad -\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle\tilde{a}^\dagger(t')\tilde{a}(t'')\tilde{\sigma}(t) - \langle\tilde{R}^+(t')\tilde{R}^-(t'')\rangle\tilde{a}(t')\tilde{a}^\dagger(t'')\tilde{\sigma}(t) \\
&\quad +\langle\tilde{R}^-(t'')\tilde{R}^+(t')\rangle\tilde{a}(t')\tilde{\sigma}(t)\tilde{a}^\dagger(t'') + \langle\tilde{R}^+(t'')\tilde{R}^-(t')\rangle\tilde{a}^\dagger(t')\tilde{\sigma}(t)\tilde{a}(t'') \\
&\quad +\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle\tilde{a}(t'')\tilde{\sigma}(t)\tilde{a}^\dagger(t') + \langle\tilde{R}^+(t')\tilde{R}^-(t'')\rangle\tilde{a}^\dagger(t'')\tilde{\sigma}(t)\tilde{a}(t') \\
&\quad -\langle\tilde{R}^-(t'')\tilde{R}^+(t')\rangle\tilde{\sigma}(t)\tilde{a}^\dagger(t'')\tilde{a}(t') - \langle\tilde{R}^+(t'')\tilde{R}^-(t')\rangle\tilde{\sigma}(t)\tilde{a}(t'')\tilde{a}^\dagger(t') \\
&\quad],
\end{aligned} \tag{23}$$

where

$$\tilde{a}(t) = \hat{a}e^{-i\omega_0 t} \tag{24}$$

$$\tilde{a}^\dagger(t) = \hat{a}^\dagger e^{i\omega_0 t} \tag{25}$$

and

$$\langle F\rangle = \mathrm{Tr}_b\left\{F\sigma_b\right\}, \tag{26}$$

for any operotor F for the environment. Here the terms involving the products $\langle\tilde{R}^+(t_1)\tilde{R}^+(t_2)\rangle$ and $\langle\tilde{R}^-(t_1)\tilde{R}^-(t_2)\rangle$ are omitted for those produce zero when evaluating the expectation values with respect to the thermal state σ_b. By putting the reduced density operator $\tilde{\sigma}(t)$ outside of the integral we have

$$\begin{aligned}\frac{\Delta\tilde{\sigma}(t)}{\Delta t} &= -\frac{1}{\Delta t}\left[\left(\hat{a}^\dagger\hat{a}\tilde{\sigma}(t)-\hat{a}\tilde{\sigma}(t)\hat{a}^\dagger\right)\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\left(\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle e^{i\omega_0\left(t'-t''\right)}\right)\right.\\ &\qquad\left.+\left(\tilde{\sigma}(t)\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\tilde{\sigma}(t)\hat{a}\right)\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\left(\langle\tilde{R}^+(t'')\tilde{R}^-(t')\rangle e^{i\omega_0\left(t'-t''\right)}\right)\right]\\ &\quad +h.c.\end{aligned} \tag{27}$$

We now calculate the two-time averages in Eq. (27). Note that from Eqs. (9) and (10) we have

$$\begin{aligned}\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}f(\omega)f^*(\omega')\underbrace{\langle\hat{c}(\omega)\hat{c}^\dagger(\omega')\rangle}_{(\langle n\rangle+1)2\pi\delta(\omega-\omega')}e^{-i\omega t'}e^{i\omega' t''}\\ &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)e^{-i\omega\left(t'-t''\right)}\end{aligned} \tag{28}$$

and similarly

$$\begin{aligned}\langle\tilde{R}^+(t'')\tilde{R}^-(t')\rangle &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}f^*(\omega)f(\omega')\underbrace{\langle\hat{c}^\dagger(\omega)\hat{c}(\omega')\rangle}_{\langle n\rangle 2\pi\delta(\omega-\omega')}e^{i\omega' t''}e^{-i\omega t'}\\ &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\langle n(\omega)\rangle e^{-i\omega\left(t'-t''\right)}\end{aligned} \tag{29}$$

where we used

$$\left[\hat{c}(\omega),\hat{c}^\dagger(\omega')\right] = 2\pi\delta(\omega-\omega') \tag{30}$$

and

$$\langle\hat{c}^\dagger(\omega)\hat{c}(\omega')\rangle = \mathrm{Tr}_b\left\{\hat{c}^\dagger(\omega)\hat{c}(\omega')\sigma_b\right\} = 2\pi\delta(\omega-\omega')\langle n(\omega)\rangle. \tag{31}$$

Since the two-time averages of $\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle$ and $\langle\tilde{R}^+(t'')\tilde{R}^-(t')\rangle$ depend only on the time difference $\tau\equiv t'-t''$ we can modify the domain of integration in Eq. (27) as shown as the shaded area in Fig. 1, that is,

$$\int_t^{t+\Delta t}dt'\int_t^{t'}dt'' \rightarrow \int_0^{\Delta t}d\tau\int_{t+\tau}^{t+\Delta t}dt'. \tag{32}$$

Now let us further assume that $\langle\tilde{R}^-(t')\tilde{R}^+(t'-\tau)\rangle$ and $\langle\tilde{R}^+(t'-\tau)\tilde{R}^-(t')\rangle$ are only non-zero within the narrow region as shown as the red-shaded area in Fig. 1 where $\tau<\tau_c\ll\Delta t$. This allows us to extend the domain of integration as shown as the shaded area in Fig. 2, i.e.,

$$\int_0^{\Delta t}d\tau\int_{t+\tau}^{t+\Delta t}dt' \rightarrow \int_0^{\infty}d\tau\int_t^{t+\Delta t}dt' \tag{33}$$

to give

$$\begin{aligned}\int_t^{t+\Delta t}dt'\int_t^{t'}dt''\left(\langle\tilde{R}^-(t')\tilde{R}^+(t'')\rangle e^{i\omega_0\left(t'-t''\right)}\right) &= \Delta t\int_0^{\infty}d\tau\left(\langle\tilde{R}^-(\tau)\tilde{R}^+(0)\rangle e^{i\omega_0\tau}\right)\\ &= \Delta t\int_0^{\infty}d\tau\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)e^{-i(\omega-\omega_0)\tau}\end{aligned} \tag{34}$$

and similarly

$$\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \left(\langle \tilde{R}^+(t'')\tilde{R}^-(t')\rangle e^{i\omega_0 \left(t'-t''\right)} \right) = \Delta t \int_0^\infty d\tau \left(\langle \tilde{R}^+(0)\tilde{R}^-(\tau)\rangle e^{i\omega_0 \tau} \right)$$
$$= \Delta t \int_0^\infty d\tau \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \langle n(\omega)\rangle e^{-i(\omega-\omega_0)\tau}. \quad (35)$$

These integrals seem to diverge at $\omega = \omega_0$. To prevent the divergence let us insert the convergence factor ϵ into Eqs. (34) and (35) and perform integrations

$$\lim_{\epsilon\to 0^+} \int_0^\infty d\tau \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \left(\langle n(\omega)\rangle + 1\right) e^{i[(\omega_0-\omega)+i\epsilon]\tau}$$
$$= i \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \left(\langle n(\omega)\rangle + 1\right) \left(-i \lim_{\epsilon\to 0^+} \int_0^\infty d\tau e^{i[(\omega_0-\omega)+i\epsilon]\tau} \right)$$
$$= i \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \left(\langle n(\omega)\rangle + 1\right) \underbrace{\left(\lim_{\epsilon\to 0^+} \frac{1}{(\omega_0-\omega)+i\epsilon} \right)}_{\mathrm{G}_+(\omega_0)} \quad (36)$$

$$\lim_{\epsilon\to 0^+} \int_0^\infty d\tau \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \langle n(\omega)\rangle e^{i[(\omega-\omega_0)+i\epsilon]\tau}$$
$$= i \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \langle n(\omega)\rangle \left(i \lim_{\epsilon\to 0^+} \int_0^\infty d\tau e^{i[(\omega_0-\omega)+i\epsilon]\tau} \right)$$
$$= i \int_{-\infty}^\infty \frac{d\omega}{2\pi} \left|f(\omega)\right|^2 \langle n(\omega)\rangle \underbrace{\left(\lim_{\epsilon\to 0^+} \frac{1}{(\omega_0-\omega)+i\epsilon} \right)}_{\mathrm{G}_+(\omega_0)}. \quad (37)$$

We now invoke the so-called *Dirac identity*,

$$G_+(\omega_0) \equiv \lim_{\epsilon\to 0^+} \frac{1}{(\omega_0-\omega)+i\epsilon} = \frac{\mathcal{P}}{\omega_0-\omega} - i\pi\delta(\omega_0-\omega) \quad (38)$$

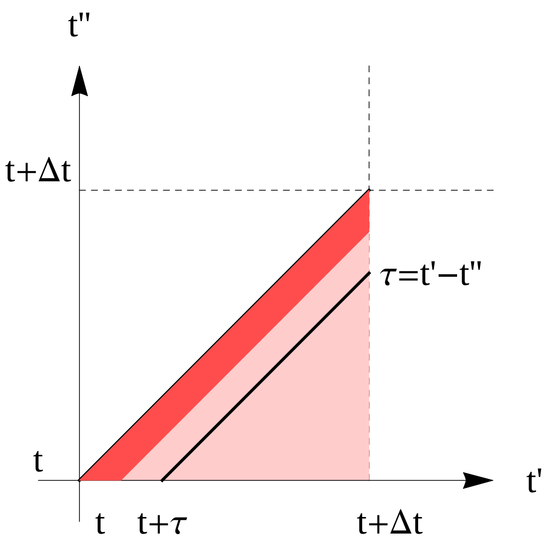

FIG. 1: Domain of integration: Eq. (32).

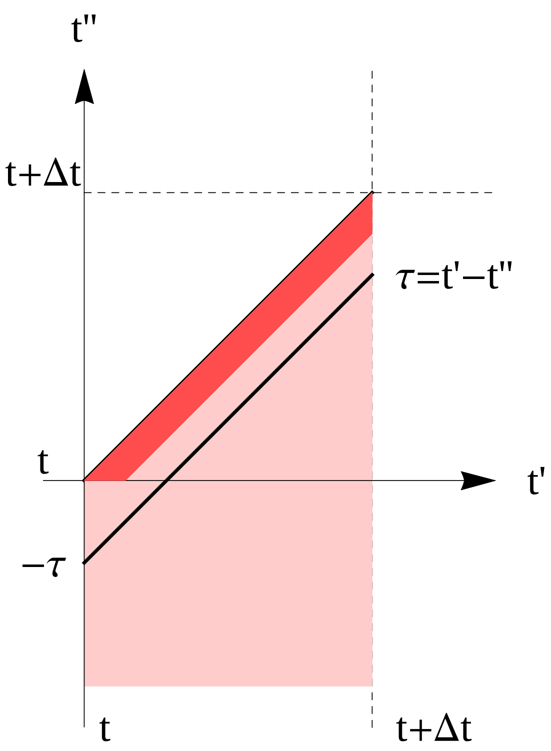

FIG. 2: Domain of integration: Eq. (33).

to get

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)G_+(\omega_0) &= i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)\left(\frac{\mathcal{P}}{\omega_0-\omega}-i\pi\delta(\omega_0-\omega)\right)\\ &= \underbrace{i\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)}{\omega_0-\omega}}_{\Delta'+\Delta}\\ &\quad+\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}d\omega\left|f(\omega)\right|^2\left(\langle n(\omega)\rangle+1\right)\delta(\omega_0-\omega)}_{\Gamma'+\Gamma}\\ &= i\left(\Delta'+\Delta\right)+\frac{\Gamma'+\Gamma}{2} \end{aligned} \tag{39}$$

$$\begin{aligned} i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\langle n(\omega)\rangle G_+(\omega_0) &= i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left|f(\omega)\right|^2\langle n(\omega)\rangle\left(\frac{\mathcal{P}}{\omega_0-\omega}-i\pi\delta(\omega-\omega_0)\right)\\ &= \underbrace{i\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\left|f(\omega)\right|^2\langle n(\omega)\rangle}{\omega-\omega_0}}_{\Delta'}\\ &\quad+\frac{1}{2}\underbrace{\int_{-\infty}^{\infty}d\omega\left|f(\omega)\right|^2\langle n(\omega)\rangle\delta(\omega-\omega_0)}_{\Gamma'}\\ &= i\Delta'+\frac{\Gamma'}{2}, \end{aligned} \tag{40}$$

where $\mathcal{P}$ is the principal part integral. For the function $F(x)$ which diverges at the point $x = x_0$, it is defined by

$$\mathcal{P}\int_a^b F(x)dx = \lim_{\delta\to 0}\left(\int_a^{x_0-\delta}F(x)dx+\int_{x_0+\delta}^{b}F(x)dx\right). \tag{41}$$

By plugging these expressions in Eq. (27) we have

$$
\begin{aligned}
\frac{\partial\tilde{\sigma}(t)}{\partial t} &= -\Bigg[\left(\hat{a}^\dagger\hat{a}\tilde{\sigma}(t)-\hat{a}\tilde{\sigma}(t)\hat{a}^\dagger\right)\left(i\left(\Delta'+\Delta\right)+\frac{\Gamma'+\Gamma}{2}\right) \\
&\quad+\left(\tilde{\sigma}(t)\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\tilde{\sigma}(t)\hat{a}\right)\left(i\Delta'+\frac{\Gamma'}{2}\right) \\
&\quad+\left(\tilde{\sigma}(t)\hat{a}^\dagger\hat{a}-\hat{a}\tilde{\sigma}(t)\hat{a}^\dagger\right)\left(-i\left(\Delta'+\Delta\right)+\frac{\Gamma'+\Gamma}{2}\right) \\
&\quad+\left(\hat{a}\hat{a}^\dagger\tilde{\sigma}(t)-\hat{a}^\dagger\tilde{\sigma}(t)\hat{a}\right)\left(-i\Delta'+\frac{\Gamma'}{2}\right)\Bigg] \\
&= -\frac{i}{\hbar}\left[\left(\hbar\Delta\hat{a}^\dagger\hat{a}\right),\tilde{\sigma}(t)\right]+\frac{\Gamma'+\Gamma}{2}\left(2\hat{a}\tilde{\sigma}(t)\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\tilde{\sigma}(t)-\tilde{\sigma}(t)\hat{a}^\dagger\hat{a}\right) \\
&\quad+\frac{\Gamma'}{2}\left(2\hat{a}^\dagger\tilde{\sigma}(t)\hat{a}-\hat{a}\hat{a}^\dagger\tilde{\sigma}(t)-\tilde{\sigma}(t)\hat{a}\hat{a}^\dagger\right).
\end{aligned}
\tag{42}
$$

Returning to the Schödinger representation we arrive at the following *master equation*:

$$
\begin{aligned}
\frac{\partial\sigma(t)}{\partial t} &= -\frac{i}{\hbar}\left[\hbar(\omega_0+\Delta)\hat{a}^\dagger\hat{a},\sigma(t)\right]+\frac{\Gamma'+\Gamma}{2}\left(2\hat{a}\sigma(t)\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\sigma(t)-\sigma(t)\hat{a}^\dagger\hat{a}\right) \\
&\quad+\frac{\Gamma'}{2}\left(2\hat{a}^\dagger\sigma(t)\hat{a}-\hat{a}\hat{a}^\dagger\sigma(t)-\sigma(t)\hat{a}\hat{a}^\dagger\right).
\end{aligned}
\tag{43}
$$

This is a *operator-valued equation*, from which the rate equation for the populations and Fokker-Planck equation for the quasi-probability density can be derived as we shall see later on.

B. Fermi's golden rule [4]

Here, let us pause for a moment and streamline what we have learned. Let us consider the situation in which an LC circuit capacitively coupled to a 1D transmission line. The interaction Hamiltonian given in Eq. (45) at the begining can be considered as

$$
\begin{aligned}
H_i &= Q_s(t)V(t) \\
&= \underbrace{\sqrt{\frac{\hbar}{2L_0\omega_0}}\left(\hat{a}+\hat{a}^\dagger\right)}_{Q_s(t)}\underbrace{\left(V^{\rightarrow}(0,t)+V^{\leftarrow}(0,t)\right)}_{V(x=0,t)} \\
&= \sqrt{\frac{\hbar}{2L_0\omega_0}}\left(\hat{a}+\hat{a}^\dagger\right)\left(-2i\int_0^\infty\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)-\hat{c}^\dagger(\omega)\right)\right),
\end{aligned}
\tag{44}
$$

where the factor "2" in the second parenthesis in the last line coming from the fact that $V^{\rightarrow}(0,t)=V^{\leftarrow}(0,t)$ for the open terminal at $x=0$. Now we invoke the *secular approximatin*, with which only the terms varing slowly with respect to the *coarse-grained time* Δt are retained, to obtain

$$
\begin{aligned}
H_i &= -i\hbar\sqrt{\frac{Z_p}{L_0}}\int_0^\infty\frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega)-\hat{a}\hat{c}^\dagger(\omega)\right) \\
&= -i\hbar\sqrt{\frac{Z_p}{L_0}}\int_{-\infty}^\infty\frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega)-\hat{a}\hat{c}^\dagger(\omega)\right),
\end{aligned}
\tag{45}
$$

where the second equation is due to the fact that the integrand is only non-zero around $\omega\sim\omega_0$ so that the domain of integration can be extended down to $-\infty$. This Hamiltonian is indeed the one we used in Eq. (4) in *note 2016-12-05* with the coupling rate $f(\omega)$ being assumed to be freqency-independent (white), that is,

$$
f(\omega)=f^*(\omega)=\sqrt{\frac{Z_p}{L_0}}=\sqrt{\Gamma}. \tag{46}
$$

This establishes the connection between the coupling constant $f(\omega)$ and the Einstein-A-coefficient like spontaneous emission rate Γ.

We can now indentify Γ defined in Eqs. (39) and (40) by

$$\Gamma = \int_{-\infty}^{\infty} d\omega \, |f(\omega)|^2 \, \delta(\omega - \omega_0) \sim |f(\omega_0)|^2 \tag{47}$$

as the Einstein-A-coefficient-like spontaneous emission (decay) rate for the LC circuit. The fact that the decay rate Γ can be obtained from the square of the coupling rate is called *Fermi's golden rule*. On the other hand Γ' defined in Eqs. (39) and (40) by

$$\Gamma' = \int_{-\infty}^{\infty} d\omega \, |f(\omega)|^2 \, \langle n(\omega)\rangle \delta(\omega - \omega_0) \sim |f(\omega_0)|^2 \, \langle n(\omega_0)\rangle = \langle n(\omega_0)\rangle \Gamma \tag{48}$$

is identified as the Einstein-B-coefficient-like stimulated emission rate (and also absorption rate).

It also becomes clear that Δ appeared in Eqs. (39) and (40) is the Lamb-shift-like *spontaneous* radiative shift occuring without any photons in the environment, and Δ' is the radiative shift due to real photons in the environment, which in fact *disappears* in Eq. (43) in a course of calculation (see Eq. (42)).

C. Lindblad form [1–3]

The master equation, Eq. (43), assumes the *Lindblad form*, which, in general, is written as

$$\begin{aligned}\frac{\partial \sigma(t)}{\partial t} &= -\frac{i}{\hbar} [\mathcal{H}, \sigma(t)] + \frac{1}{2} \sum_J \left(2A_J \sigma(t) A_J^\dagger - A_J^\dagger A_J \sigma(t) - \sigma(t) A_J^\dagger A_J \right) \\ &= -\frac{i}{\hbar} [\mathcal{H}, \sigma(t)] + \frac{1}{2} \sum_J \mathcal{L}_D[A_J] \, \sigma(t),\end{aligned} \tag{49}$$

with $\mathcal{H}$ being some hermitial operator and A_J being any operators (called *collapse operators*), which are responsible for the irreversible non-unitary evolution of $\sigma(t)$. The Lindblad form, however, assures to preserve the complete positivity and trace of the reduced density operator $\sigma(t)$. Here the *Lindblad superoperator* $\mathcal{L}_D[A_J] \, \sigma(t)$ is defined by

$$\mathcal{L}_D[A_J] \, \sigma(t) = 2A_J \sigma(t) A_J^\dagger - A_J^\dagger A_J \sigma(t) - \sigma(t) A_J^\dagger A_J. \tag{50}$$

The master equation (43) is then written with the Lindblad superoperators as

$$\frac{\partial \sigma(t)}{\partial t} = -\frac{i}{\hbar} \left[\hbar(\omega_0 + \Delta)\hat{a}^\dagger \hat{a}, \sigma(t)\right] + \frac{1}{2}\mathcal{L}_D\Big[\sqrt{\Gamma' + \Gamma}\hat{a}\Big] \, \sigma(t) + \frac{1}{2}\mathcal{L}_D\Big[\sqrt{\Gamma'}\hat{a}^\dagger\Big] \, \sigma(t). \tag{51}$$

III. RATE EQUATION

A. Rate equation [1]

Now we shall see more explicitly that $\Gamma = |f(\omega_0)|^2$ in Eq. (47) is related to the Einstein-A-coefficient-like spontaneous emission rate. Let us first evaluate the master equation (43) in the Fock-state bases $\{|N\rangle\}$ for the system:

$$\begin{aligned}\frac{dP(N,t)}{dt} &= \frac{\Gamma + \Gamma'}{2} \left(2\langle N|\hat{a}\sigma(t)\hat{a}^\dagger|N\rangle - \langle N|\hat{a}^\dagger \hat{a}\sigma(t)|N\rangle - \langle N|\sigma(t)\hat{a}^\dagger \hat{a}|N\rangle\right) \\ &\qquad + \frac{\Gamma'}{2} \left(2\langle N|\hat{a}^\dagger\sigma(t)\hat{a}|N\rangle - \langle N|\hat{a}\hat{a}^\dagger \sigma(t)|N\rangle - \langle N|\sigma(t)\hat{a}\hat{a}^\dagger|N\rangle\right) \\ &= \underbrace{\Gamma_\uparrow}_{\Gamma'} NP(N-1) - \underbrace{\Gamma_\uparrow}_{\Gamma'} (N+1)P(N) - \underbrace{\Gamma_\downarrow}_{\Gamma+\Gamma'} NP(N) + \underbrace{\Gamma_\downarrow}_{\Gamma+\Gamma'} (N+1)P(N+1),\end{aligned} \tag{52}$$

where we used the abbreviations $P(N,t) = \langle N|\sigma(t)|N\rangle$, $P(N-1,t) = \langle N-1|\sigma(t)|N-1\rangle$, and $P(N+1,t) = \langle N+1|\sigma(t)|N+1\rangle$. Here the first term in the right hand side of Eq. (43) naturally disappers since it represents the conservative Hamiltonian and thus contributes nothing to the dynamics of Fock states (energy eigenstates). Consequently, this equation contains only the diagonal elements of the density operator and the effect of the radiative shifts Δ and Δ' are absent. There are 4 terms in Eq. (52), each of which has a clear physical meaning as depicted in

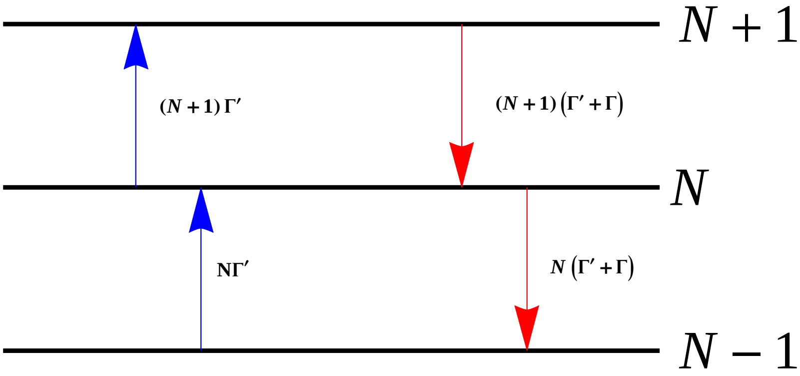

FIG. 3: Physicsl interpretations of 4 terms in Eq. (52).

Fig. 3. The first term represents the population gain from the $(N-1)$ −photon Fock state due to the absorption at the rate of $N\Gamma'$ (the lower blue arrow in Fig. 3). The second term represents the population loss from the $N-$photon Fock state due to the absorption at the rate of $(N+1)\,\Gamma'$ (the upper blue arrow in Fig. 3). The third and the fourth terms represent the spontaneous and stimulated emissions from the $N-$photon Fock state into the $(N-1)-$photon Fock state at the rate of $N\,(\Gamma'+\Gamma)$ (the lower red arrow in Fig. 3) and from the $(N+1)-$photon Fock state into the $N-$photon Fock state at the rate of $(N+1)\,(\Gamma'+\Gamma)$ (the upper red arrow in Fig. 3), respectively.

We have then the following rate equation:

$$\begin{aligned}
\frac{d}{dt}\langle N\rangle &\equiv \sum_N N\frac{P(N,t)}{dt} \\
&= \Gamma_\uparrow \sum_N (N+1)P(N) - \Gamma_\downarrow \sum_N NP(N), \\
&= \Gamma_\uparrow \langle N+1\rangle - \Gamma_\downarrow \langle N\rangle \\
&= -\left(\Gamma_\downarrow - \Gamma_\uparrow\right)\langle N\rangle + \Gamma_\uparrow \\
&= -\Gamma\langle N\rangle + \Gamma' \\
&= -\Gamma\langle N\rangle + \Gamma\langle n(\omega_0)\rangle,
\end{aligned} \tag{53}$$

where Eq. (48) is used in the last equation. Thus, as we anticipated, Γ can be considered as the spontaneous emission rate. Note that the bosonic enhancement factors are canceled to get the damping rate independent on the initial excitations. This cancellation stems form the fact that the harmonic oscillators have an equidistant evergy-level structure. The steady state condition $\frac{d}{dt}\langle N\rangle = 0$ gives us the steady-state occupation number of the LC photons:

$$\bar{N} = \frac{\Gamma'}{\Gamma} = \langle n(\omega_0)\rangle, \tag{54}$$

that is, the averaged photon number of the environment $\langle n(\omega_0)\rangle$.

The general solution of the rate equation Eq. (58) can be given by

$$\langle N(t)\rangle = \langle N(0)\rangle e^{-\Gamma t} + \langle n(\omega_0)\rangle\left(1-e^{-\Gamma t}\right). \tag{55}$$

B. Detailed balance [4]

We identify $\Gamma_\downarrow$ and $\Gamma_\uparrow$ in Eq. (52) as

$$\begin{aligned}\Gamma_\downarrow = \Gamma' + \Gamma &= |f(\omega_0)|^2 \left(\langle n(\omega_0)\rangle + 1\right) \\ &= \frac{Z_p}{L_0}\left(\langle n(\omega_0)\rangle + 1\right) \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(\underbrace{2Z_p\hbar\omega_0\left(\langle n(\omega_0)\rangle + 1\right)}_{S_{VV}(\omega_0)}\right) \\ &= \frac{1}{2\hbar\omega_0 L_0} S_{VV}(\omega_0),\end{aligned} \tag{56}$$

and

$$\begin{aligned}\Gamma_\uparrow = \Gamma' &= |f(\omega_0)|^2 \langle n(\omega_0)\rangle \\ &= \frac{Z_p}{L_0}\langle n(\omega_0)\rangle \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(\underbrace{2Z_p\hbar\omega_0\langle n(\omega_0)\rangle}_{S_{VV}(-\omega_0)}\right) \\ &= \frac{1}{2\hbar\omega_0 L_0} S_{VV}(-\omega_0).\end{aligned} \tag{57}$$

In words, the downward decay rate $\Gamma_\downarrow$ and the upward decay rate $\Gamma_\uparrow$ are related to the noise spectral densities $S_{VV}(\omega_0)$ and $S_{VV}(-\omega_0)$, respectively.

Now the rate equation (see Eq. (58) suggests that

$$\begin{aligned}\frac{d}{dt}\langle N\rangle &= \Gamma_\uparrow\langle N+1\rangle - \Gamma_\downarrow\langle N\rangle \\ &= \frac{1}{2\hbar\omega_0 L_0}\left(S_{VV}(-\omega_0)\langle N+1\rangle - S_{VV}(-\omega_0)\langle N\rangle\right).\end{aligned} \tag{58}$$

Note that N stands for the photon number in the LC circuit and $n(\omega_0)$ stands for the photon number in the 1D transmission line at angular frequency ω_0. In the steady state we have $\frac{d}{dt}\langle N\rangle = 0$ and thus

$$\begin{aligned}\frac{\Gamma_\uparrow}{\Gamma_\downarrow} &= \frac{S_{VV}(-\omega_0)}{S_{VV}(\omega_0)} \\ &= \frac{\langle N\rangle}{\langle N+1\rangle} = \frac{\frac{1}{e^{\frac{\hbar\omega_0}{k_B T}}-1}}{\frac{1}{e^{\frac{\hbar\omega_0}{k_B T}}-1}+1} = e^{-\frac{\hbar\omega_0}{k_B T}},\end{aligned} \tag{59}$$

where the thermal equilibrium is assumed. Equation (59) is called the *detailed balance condtion*, suggesting that the asymmetry of the noise power spectral densities $S_{VV}(-\omega_0)$ and $S_{VV}(\omega_0)$ are related to the temperature T. This fact was used before when we deduce the classical Nyquist formula from the quantum counterpart.

Appendix A: Fokker-Planck equation [1]

In Sec. III we evaluate the master equation (43) in the Fock-state bases $\{|N\rangle\}$ to obtain the rate equation. When we evaluate the master equation (43) in the coherent-state bases $\{|\beta\rangle\}$, we arrive at the *Fokker-Planck equation* for the quasi-probability density called *P-function*, $P_N(\beta,\beta^*,t)$. Here the P-function can be used to represent the density operator $\sigma(t)$ as

$$\sigma(t) = \int d^2\beta P_N(\beta,\beta^*,t)|\beta\rangle\langle\beta|, \tag{A1}$$

where the coherent-state bases $\{|\beta\rangle\}$ are defined as

$$\hat{a}|\beta\rangle = \beta|\beta\rangle \tag{A2}$$

$$\langle\beta|\hat{a}^\dagger = \beta^*\langle\beta|. \tag{A3}$$

From Eq. (43) with Eq. (A1), we have the Fokker-Planck equation:

$$\begin{aligned}\frac{\partial}{\partial t}P_N(\beta,\beta^*,t) &= \left(\frac{\Gamma}{2}+i\left(\omega_0+\Delta\right)\right)\frac{\partial}{\partial\beta}\left[\beta P_N(\beta,\beta^*,t)\right] \\ &+\left(\frac{\Gamma}{2}-i\left(\omega_0+\Delta\right)\right)\frac{\partial}{\partial\beta^*}\left[\beta^* P_N(\beta,\beta^*,t)\right] \\ &+\Gamma'\frac{\partial^2}{\partial\beta\partial\beta^*}P_N(\beta,\beta^*,t).\end{aligned} \tag{A4}$$

The following Gaussian function

$$P_N(\beta,\beta,t) = \frac{1}{\pi\langle n(\omega_0)\rangle\left(1-e^{-\Gamma t}\right)}\exp\left(-\frac{\left[\beta-\beta_0 e^{-\left(\frac{\Gamma}{2}+i(\omega_0+\Delta)t\right)}\right]\left[\beta^*-\beta_0^* e^{-\left(\frac{\Gamma}{2}-i(\omega_0+\Delta)t\right)}\right]}{\langle n(\omega_0)\rangle\left(1-e^{-\Gamma t}\right)}\right) \tag{A5}$$

constitutes the solution of Eq. (A4). Note that the function $P_N(\beta,\beta^*,t)$ becomes $\delta(\beta-\beta_0)\delta(\beta^*-\beta_0^*)$ at $t=0$ and then shows a diffusion-like behavior.

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