Path integral for spin

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To better understand the more wide variety of topological quantum phenomena we shall crank up the level of abstraction. In particular we shall here look at the path integral for spin, who lives in a space where the *Lie group SU(2)* can act, a kind of differential manifold. Understanding this strange path integral equips us with enough background to appreciate the Berry phase and the related topological phenomena in quantum mechanics.

I. FROM CONFIGURATION SPACE TO LIE GROUP SU(2) [1, 2]

The Hilbert space of a spin-$\frac{1}{2}$ particle with the kinematic degrees of freedom frozen is spaned by $|\uparrow\rangle$ and $|\downarrow\rangle$. Each state in this Hilbert space can be obtained by applying a group element g of Lie group SU(2) on $|\uparrow\rangle$. SU(2) stands for *special unitary group in two dimentions.* g can be parameterized by three *Euler angles*, ϕ, θ, and ψ:

$$g(\phi,\theta,\psi) = e^{-i\phi\sigma_3}e^{-i\theta\sigma_2}e^{-i\psi\sigma_3}, \tag{1}$$

where

$$\sigma_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

$$\sigma_2 = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{3}$$

$$\sigma_3 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4}$$

are the *generators*, which are the *Pauli matrices.* The three *Euler angles*, ϕ, θ, and ψ, can be viewed as the position q in the 1D configuration space or x, y, and z in the 3D configuration space. SU(2) can thus be viewed as a *manifold* whose tangent space is called the Lie algebra $su(2)$, which consists of the Pauli matrices with the commutation relation:

$$[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k. \tag{5}$$

SU(2) has a natural measure, the so-called *Haar measure*, which make it possible to integrate g over SU(2). The integral is defined like

$$\int_{\mathrm{SU(2)}} dg f(g) \tag{6}$$

where $f(g)$ is some function of g.

II. PATH INTEGRAL FOR SPIN [2]

With these rudimentary background on SU(2), let us construct the path integral for spin. What we want to calculate here is the partition function

$$\mathcal{Z} = \mathrm{tr}e^{-\beta H} = \int_{\mathrm{SU(2)}} dg \langle g|e^{-\beta H}|g\rangle \tag{7}$$

with $\beta = \frac{1}{k_{\mathrm{B}}T}$ being the *inverse temperature* and H being the Zeeman Hamiltonian,

$$H = -\underbrace{\boldsymbol{m}}_{-\hbar\gamma_s\boldsymbol{\sigma}} \cdot \boldsymbol{B}, \tag{8}$$

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where $\boldsymbol{m}$ is the magnetization, $\boldsymbol{B}$ is the magnetic field and $\gamma_s = \frac{g\mu_B}{\hbar}$ is the *gyromagnetic ratio* with g being the g-factor of the electron spin and μ_B being the *Bohr magneton*. Here γ_s can be considered as the ratio: (magnetic moment)/(angular momentum). Here $|g\rangle$ stands for $g|\uparrow\rangle$. The form of $\mathcal{Z}$ in Eq. (7) can be recognized to be very similar to the transition probability amplitude

$$\langle q_f, t|q_i, 0\rangle = \langle q_f| \exp\left[-\frac{i}{\hbar}Ht\right] |q_i\rangle. \tag{9}$$

The partition function $\mathcal{Z}$ can thus be obtained through a kind of path integral with imaginary time $\beta = \frac{it}{\hbar}$, which is called *Euclidean path integral*.

Let us repeat the same procedure to reach the Feynman path integral by replacing the time t with the abstruct *imaginary time* β. The first step is to chop the imaginary time interval β into N slices τ ($\beta = N\tau$) we have

$$\begin{aligned} \langle g|e^{-\beta H}|g\rangle &= \langle g|\underbrace{e^{-\tau H}}_{1}\underbrace{e^{-\tau H}}_{2}\cdots\underbrace{e^{-\tau H}}_{N}|g\rangle \\ &\simeq \langle g|\underbrace{(1-\tau H)}_{1}\underbrace{(1-\tau H)}_{2}\cdots\underbrace{(1-\tau H)}_{N}|g\rangle. \end{aligned} \tag{10}$$

By introducing the resolution of identity,

$$I = \int dg|g\rangle\langle g|, \tag{11}$$

into each slice the partition function becomes

$$\mathcal{Z} = \int_{g_0=g_N} \prod_{i=0}^{N} dg_i \exp\left[-\tau\sum_{i=0}^{N-1}\left(-\frac{\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle}{\tau} + \langle g_{i+1}|H|g_i\rangle\right)\right], \tag{12}$$

where we used

$$\begin{aligned} \langle g_{i+1}|\,(1-\tau H)\,|g_i\rangle &= \langle g_{i+1}|g_i\rangle - \tau\langle g_{i+1}|H|g_i\rangle \\ &= \underbrace{1-\langle g_i|g_i\rangle}_{0} + \langle g_{i+1}|g_i\rangle - \tau\langle g_{i+1}|H|g_i\rangle \\ &= \exp\left[\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle - \tau\langle g_{i+1}|H|g_i\rangle\right]. \end{aligned} \tag{13}$$

By taking the continuum limit $N \to \infty$ with

$$\tau\sum_{i=0}^{N-1} \to \int_0^\beta d\tau \tag{14}$$

$$\frac{\langle g_{i+1}|g_i\rangle - \langle g_i|g_i\rangle}{\tau} \to \left\langle \frac{\partial}{\partial\tau}g(\tau)\middle| g(\tau)\right\rangle, \tag{15}$$

we have

$$\mathcal{Z} = \int Dg \exp\left[-\int_0^\beta d\tau\left(-\left\langle\frac{\partial}{\partial\tau}g(\tau)\middle|g(\tau)\right\rangle + \langle g(\tau)|H|g(\tau)\rangle\right)\right]. \tag{16}$$

This is an abstract form of the path integral for spin. Here, we have $\mathcal{Z}$ as the (g−τ)-integral, i.e., the SU(2)-inverse temperature integral, instead of the (q−t)-integral, i.e., the space-time integral!

A. Euler angle representation [2]

To get more down-to-earth insignt into the path integral for spin, let us use the Euler angles defined in Eq. (1). A state $|g(\tau)\rangle$ can then be given by

$$\begin{aligned} |g(\tau)\rangle &= e^{-i\phi(\tau)\sigma_3}e^{-i\theta(\tau)\sigma_2}e^{-i\psi(\tau)\sigma_3}|\uparrow\rangle \\ &= \left(e^{-i\frac{1}{2}\psi(\tau)}\right)e^{-i\phi(\tau)\sigma_3}e^{-i\theta(\tau)\sigma_2}|\uparrow\rangle, \end{aligned} \tag{17}$$

since $\sigma_3|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$. The state $|g(\tau)\rangle$ given by Eq. (17) is called *coherent spin state* living on the 2-sphere S^2, which is parameterized by $\phi(\tau)$ and $\theta(\tau)$, with *U(1) gauge* factor factor $e^{-i\frac{1}{2}\psi(\tau)}$. Note that since $\langle g|e^{-\beta H}|g\rangle$ is what we shall evaluate the initial state $|g(0)\rangle$ and the final state $|g(\beta)\rangle$ in the path integral are the same state, i.e., we have

$$\begin{aligned}\phi(0) &= \phi(\beta) && (18)\\ \theta(0) &= \theta(\beta) && (19)\\ \psi(0) &= \psi(\beta). && (20)\end{aligned}$$

The second term in the path integral, Eq. (16), can then be evaluated as

$$S_0[\phi,\theta] \equiv \int_0^\beta d\tau \langle g|H|g\rangle = \hbar\gamma_s \int_0^\beta d\tau \langle\uparrow|e^{i\theta\sigma_2}e^{i\phi\sigma_3}\left(\boldsymbol{\sigma}\cdot\boldsymbol{B}\right)e^{-i\phi\sigma_3}e^{-i\theta\sigma_2}|\uparrow\rangle. \tag{21}$$

Here note that the sheer phase factor $e^{-i\frac{1}{2}\psi}$ appeared in Eq. (17) disappears in Eq. (21). Now, without loss of generality, suppose that the magnetic field is pointing positive z direction, that is,

$$\boldsymbol{B} = B\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{22}$$

Then $S_0[\phi,\theta]$ becomes

$$\begin{aligned}S_0[\phi,\theta] &= \hbar\gamma_s B\int_0^\beta d\tau\langle\uparrow|e^{i\theta\sigma_2}e^{i\phi\sigma_3}\sigma_3 e^{-i\phi\sigma_3}e^{-i\theta\sigma_2}|\uparrow\rangle\\ &= \hbar\gamma_s B\int_0^\beta d\tau\langle\uparrow|e^{i\theta\sigma_2}\sigma_3 e^{-i\theta\sigma_2}|\uparrow\rangle. \qquad (23)\end{aligned}$$

The term, $e^{i\theta\sigma_2}\sigma_3e^{-i\theta\sigma_2}$, can be calculated by the following relation (a variant of Baker-Campbell-Hausdorff formula):

$$\begin{aligned}e^{i\theta\sigma_2}\sigma_3e^{-i\theta\sigma_2} &= \left(1+i\theta\sigma_2-\frac{1}{2}\theta^2\sigma_2^2-\frac{i}{6}\theta^3\sigma_2^3\cdots\right)\sigma_3\left(1-i\theta\sigma_2-\frac{1}{2}\theta^2\sigma_2^2+\frac{i}{6}\theta^3\sigma_2^3\cdots\right)\\ &= \sigma_3+i\theta\underbrace{[\sigma_2,\sigma_3]}_{i\sigma_1}-\frac{1}{2}\theta^2\underbrace{[\sigma_2,[\sigma_2,\sigma_3]]}_{\sigma_3}-\frac{i}{6}\theta^3\underbrace{[\sigma_2,[\sigma_2,[\sigma_2,\sigma_3]]]}_{i\sigma_1}\cdots\\ &= \sigma_3-\theta\sigma_1-\frac{1}{2}\theta^2\sigma_3+\frac{1}{6}\theta^3\sigma_1\cdots\\ &= \left(1-\frac{1}{2}\theta^2+\cdots\right)\sigma_3-\left(\theta-\frac{1}{6}\theta^3+\cdots\right)\sigma_1\\ &= \cos\theta\sigma_3-\sin\theta\sigma_1. \qquad (24)\end{aligned}$$

Thus, we finally have

$$\begin{aligned}S_0[\phi,\theta] &= \hbar\gamma_s B\int_0^\beta d\tau\langle\uparrow|\left(\cos\theta\sigma_3-\sin\theta\sigma_1\right)|\uparrow\rangle\\ &= \hbar\gamma_s B\int_0^\beta d\tau\left(\cos\theta\underbrace{\langle\uparrow|\sigma_3|\uparrow\rangle}_{\frac{1}{2}}-\sin\theta\underbrace{\langle\uparrow|\sigma_1|\uparrow\rangle}_{0}\right)\\ &= \frac{1}{2}\hbar\gamma_s B\int_0^\beta d\tau\cos\theta. \qquad (25)\end{aligned}$$

The first term in the path integral, Eq. (16), can be similarly evaluated:

$$
\begin{aligned}
S_{\rm top}[\phi,\theta] &\equiv -\int_0^\beta d\tau \left\langle \frac{\partial}{\partial\tau} g \middle| g \right\rangle \\
&= -\int_0^\beta d\tau \left\{ \frac{\partial}{\partial\tau} \left(e^{i\frac{1}{2}\psi} \langle\uparrow| e^{i\theta\sigma_2} e^{i\phi\sigma_3} \right) \right\} \left\{ e^{-i\frac{1}{2}\psi} e^{-i\phi\sigma_3} e^{-i\theta\sigma_2} |\uparrow\rangle \right\} \\
&= -\underbrace{\int_0^\beta d\tau \left(i\frac{1}{2}\frac{\partial\psi}{\partial\tau} \right) \langle\uparrow|\uparrow\rangle}_{i\frac{1}{2}\int_0^\beta d\tau \frac{\partial\psi}{\partial\tau} = i\frac{1}{2}(\psi(\beta)-\psi(0))=0} - \int_0^\beta d\tau \left\{ \frac{\partial}{\partial\tau} \left(\langle\uparrow| e^{i\theta\sigma_2} e^{i\phi\sigma_3} \right) \right\} \left\{ e^{-i\phi\sigma_3} e^{-i\theta\sigma_2} |\uparrow\rangle \right\} \\
&= -\int_0^\beta d\tau \left\{ \frac{\partial}{\partial\tau} \left(\langle\uparrow| e^{i\theta\sigma_2} e^{i\phi\sigma_3} \right) \right\} \left\{ e^{-i\phi\sigma_3} e^{-i\theta\sigma_2} |\uparrow\rangle \right\}.
\end{aligned} \tag{26}
$$

Note again that the sheer phase factor $e^{-i\frac{1}{2}\psi}$ appeared in Eq. (17) disappears in Eq. (26) too. Equations (21) and (26) reassure that the Euler representation of the actions $S_0[\phi,\theta]$ and $S_{\rm top}[\phi,\theta]$ are independent on the *U(1) phase* ψ of $|g\rangle$ and are thus *gauge-invariant*!

By using the similar trick as used in Eq. (24), $S_{\rm top}[\phi,\theta]$ in Eq. (26) can be given by

$$
\begin{aligned}
S_{\rm top}[\phi,\theta] &= -\int_0^\beta d\tau \left\{ \frac{\partial}{\partial\tau} \left(\langle\uparrow| e^{i\theta\sigma_2} e^{i\phi\sigma_3} \right) \right\} \left\{ e^{-i\phi\sigma_3} e^{-i\theta\sigma_2} |\uparrow\rangle \right\} \\
&= -\int_0^\beta d\tau \left(\langle\uparrow| \left(i\sigma_2 \frac{\partial\theta}{\partial\tau} \right) |\uparrow\rangle + \langle\uparrow| \left(e^{i\theta\sigma_2} i\sigma_3 \frac{\partial\phi}{\partial\tau} e^{-i\theta\sigma_2} \right) |\uparrow\rangle \right) \\
&= -\int_0^\beta d\tau \left(i\frac{\partial\theta}{\partial\tau} \underbrace{\langle\uparrow|\sigma_2|\uparrow\rangle}_{0} + i\frac{\partial\phi}{\partial\tau} \underbrace{\langle\uparrow| \left(e^{i\theta\sigma_2}\sigma_3 e^{-i\theta\sigma_2} \right) |\uparrow\rangle}_{\langle\uparrow|(\cos\theta\sigma_3 - \sin\theta\sigma_1)|\uparrow\rangle} \right) \\
&= -i\int_0^\beta d\tau \frac{\partial\phi}{\partial\tau} \left(\cos\theta \underbrace{\langle\uparrow|\sigma_3|\uparrow\rangle}_{\frac{1}{2}} - \sin\theta \underbrace{\langle\uparrow|\sigma_1|\uparrow\rangle}_{0} \right) \\
&= -\frac{i}{2}\int_0^\beta d\tau \frac{\partial\phi}{\partial\tau}\cos\theta \\
&= \frac{i}{2}\int_0^\beta d\tau \frac{\partial\phi}{\partial\tau}(1-\cos\theta),
\end{aligned} \tag{27}
$$

where the last manipulation is valid since

$$
\int_0^\beta d\tau \frac{\partial\phi}{\partial\tau} = \phi(\beta) - \phi(0) = 0. \tag{28}
$$

The action $S_{\rm top}[\phi,\theta]$ of Eq. (27) is called the *Berry phase action*, for which we shall explore in detail later on.

With Eqs. (25) and (27) we arrive at the Euler angle representation of the path integral of spin given in Eq. (16):

$$
\begin{aligned}
\mathcal{Z} &= \int Dq \exp\left[-S[\phi,\theta]\right] \\
&= \int Dq \exp\left[-\left(S_0[\phi,\theta] + S_{\rm top}[\phi,\theta]\right)\right] \\
&= \int Dq \exp\left[-\int_0^\beta d\tau \underbrace{\left(\frac{1}{2}\hbar\gamma_s B\cos\theta + \frac{i}{2}(1-\cos\theta)\dot\phi \right)}_{L(\phi,\dot\phi,\theta,\dot\theta)} \right].
\end{aligned} \tag{29}
$$

III. LARMOR PRECESSION [2, 3]

Let us now see the Euler-Lagrange equation drived from the action $S[\phi,\theta]$ in Eq. (29) is meaningful. The Lagrangian for the current problem can be read from Eq. (29) as

$$\mathcal{L}(\phi,\dot{\phi},\theta,\dot{\theta}) = \frac{1}{2}\hbar\gamma_s B\cos\theta + \frac{i}{2}\left(1-\cos\theta\right)\dot{\phi} \tag{30}$$

The Euler-Lagrange equations are thus

$$\frac{\partial\mathcal{L}(\phi,\theta)}{\partial\phi} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}(\phi,\theta)}{\partial\dot{\phi}}\right) = 0 \;\Rightarrow\; -\frac{d}{d\tau}\left\{i\left(1-\cos\theta\right)\right\} = 0 \;\Rightarrow\; \dot{\theta} = 0 \tag{31}$$

$$\frac{\partial\mathcal{L}(\phi,\theta)}{\partial\theta} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}(\phi,\theta)}{\partial\dot{\theta}}\right) = 0 \;\Rightarrow\; -\hbar\gamma_s B\sin\theta + i\sin\theta\dot{\phi} = 0 \;\Rightarrow\; \dot{\phi} = -i\hbar\gamma_s B. \tag{32}$$

Note, however that this action is the *imaginary action* with the *imaginary time* $\tau = \frac{it}{\hbar}$. The real-time Euler-Lagrange equations could read

$$\dot{\theta} = 0 \tag{33}$$

$$\dot{\phi} = \gamma_s B. \tag{34}$$

We shall now check that these two equations indeed describe *Larmor precession* of a magnetic moment of the electron spin

$$\boldsymbol{m} = -\mathrm{g}\mu_{\mathrm{B}}\boldsymbol{\sigma} = -\hbar\gamma_s\boldsymbol{\sigma} \tag{35}$$

in the magnetic field $\boldsymbol{B}$. Let the magnetic moment $\boldsymbol{m}$ be written in terms of two Euler angles as

$$\boldsymbol{m} = m_0\begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix}, \tag{36}$$

where $m_0 = \frac{\mathrm{g}\mu_{\mathrm{B}}}{2}$ and we define the spherical orthonormal system as

$$\boldsymbol{e}_r = \begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix} \tag{37}$$

$$\boldsymbol{e}_\theta = \begin{bmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{bmatrix} \tag{38}$$

$$\boldsymbol{e}_\phi = \begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\end{bmatrix}. \tag{39}$$

The equation of motion for the magnetic moment $\boldsymbol{m}$ in the magnetic field $\boldsymbol{B} = \begin{bmatrix}0\\0\\B\end{bmatrix}$ can then be written by

$$\dot{\boldsymbol{m}} = -\gamma_s\boldsymbol{m}\times\boldsymbol{B}. \tag{40}$$

This equation of motion as can be derived from the following argument: The angular momentum of the magnetic moment is given by $-\dfrac{\boldsymbol{m}}{\gamma_s} = \hbar\boldsymbol{\sigma}$. The rate of the change of the angular momentum, $\hbar\dot{\boldsymbol{\sigma}} = -\dfrac{\dot{\boldsymbol{m}}}{\gamma_s}$ is equal to the torque $\boldsymbol{\mathcal{T}}$. The torque here is given by

$$\boldsymbol{\mathcal{T}} = \boldsymbol{m}\times\boldsymbol{B}. \tag{41}$$

Thus we have

$$-\frac{\dot{\boldsymbol{m}}}{\gamma_s} = \boldsymbol{m}\times\boldsymbol{B}, \tag{42}$$

which is indeed equal to Eq. (40).

In terms of Euler angles, Eq. (40) becomes the following form:

$$m_0\dot{\theta}\begin{bmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{bmatrix}+m_0\sin\theta\dot{\phi}\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\end{bmatrix}=\gamma_s Bm_0\sin\theta\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\end{bmatrix}, \tag{43}$$

which can be rewritten as

$$m_0\dot{\theta}\boldsymbol{e}_\theta+m_0\sin\theta\dot{\phi}\boldsymbol{e}_\phi=\gamma_s Bm_0\sin\theta\boldsymbol{e}_\phi. \tag{44}$$

Equation (44) is indeed equivalent to the two Euler-Lagrange equations Eqs. (33) and (34).

This procedure of deriving the Larmor precession from the Lagrangian Eq. (30) reminds us of the procedure of deriving the Lorentz force from the guage-dependent Lagrangian

$$L(\boldsymbol{x},\dot{\boldsymbol{x}})=\frac{1}{2}m\dot{\boldsymbol{x}}^2+q\boldsymbol{A}\cdot\dot{\boldsymbol{x}} \tag{45}$$

in the last lecture. This resemblance is even more striking when $L(\boldsymbol{x},\dot{\boldsymbol{x}})$ is representing with imaginary time $t\to -i\tau$, that is,

$$\mathcal{L}(\boldsymbol{x},\dot{\boldsymbol{x}})=\frac{1}{2}m\dot{\boldsymbol{x}}^2+iq\boldsymbol{A}\cdot\dot{\boldsymbol{x}}, \tag{46}$$

which can be seen to be similar to the Euclidean Lagrangian $\mathcal{L}(\phi,\dot{\phi},\theta,\dot{\theta})$ given in Eq. (30). We shall explore this resemblance further in the next lecture.

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[3] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinenann, Oxford, England, 1975).