Quantum Hall effect

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(Dated: October 25, 2021)

In 1980, von Klitzing, Dorda, and Pepper reported [1] an experiment which measured the quantized Hall resistance of the value $R_{\mathrm{H}} = 6453.3 \pm 0.1\ \Omega$ for a 2D electron gas under the strong magnetic field. We shall explore this *quantum Hall effect* in the light of Berry phase.

I. THE DISCOVERY [1]

In 1980, von Klitzing, Dorda, and Pepper reported [1] that under the strong magnetic field the Hall resistance shows a plateau at the value

$$R_{\mathrm{H}} = 6453.3 \pm 0.1\ \Omega \tag{1}$$

for measurements in which the Fermi level lies within the energy gap between the Landau quantization levels in a 2D electron gas as shown in Fig. 1.

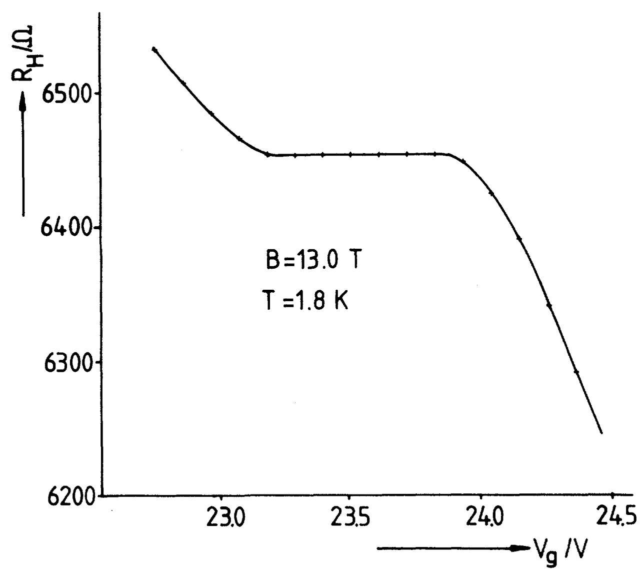

FIG. 1. Experimentally measured Hall resistance R_{H} as a function of gate voltage V_{g} under the strong magnetic field [1]. When the gate voltage was in a region corresponding to the energy gap between the lowest and the first excited Landau quantization levels in a 2D electron gas, there is a plateau in R_{H}. The plateau has a value of $6453.3 \pm 0.1\ \Omega$.

How can this precise quantized value of the Hall resistance be possible? We shall now explore this *quantum Hall effect* in the light of Berry phase.

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II. MAGNETIC BLOCH BAND [2–4]

A. Bloch electron in a uniform magnetic field

Let us consider an electron in a 2D crystal laying xy-plane under the strong magnetic field $\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$ perpendicular to the $xy-$plane. The Schrödinger equation for the electrons is written as

$$H_B(\boldsymbol{r})\,|\psi(\boldsymbol{r})\rangle = \left[\frac{1}{2m}\left(\{\boldsymbol{p}+e\boldsymbol{A}(\boldsymbol{r})\}^2 + U(\boldsymbol{r})\right)\right]|\psi(\boldsymbol{r})\rangle = E\,|\psi(\boldsymbol{r})\rangle\,, \tag{2}$$

where the position and momentum of the electron are confined in xy-plane within the uncertainty relation. We set $\boldsymbol{r} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ and $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla} = -i\hbar \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$. We use the so-called *symmetric gauge* for the vector potential, which can be written by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}\begin{bmatrix} yB \\ -xB \\ 0 \end{bmatrix}, \tag{3}$$

and found within $xy-$plane. The potential $U(\boldsymbol{r}) = U(x,y)$ is periodic in both $x-$ and $y-$directions, that is,

$$U(x+a,y) = U(x,y+b) = U(x,y), \tag{4}$$

where a and b are the lattice constants along $x-$ and $y-$directions, respectively.

We shall now ask: can we simply employ the Bloch states as the solutions of Eq. (2)?

No!

Although the perpendicular magnetic field $\boldsymbol{B}$ is uniform over the 2D crystal, the Hamiltonian is not invariant under the spatial translations $\boldsymbol{r} \to \boldsymbol{r}+\boldsymbol{R}$, that is, $x \to x+na$ and $y \to y+mb$ with $\boldsymbol{R} = \begin{bmatrix} na \\ mb \\ 0 \end{bmatrix}$. In other words, the presence of the magnetic field $\boldsymbol{B}$ induce the presence of the spatially-non-uniform vector potential $\boldsymbol{A}(\boldsymbol{r})$ in the Hamiltonian $H_B(\boldsymbol{r})$ in Eq. (2), breaking the translational symmetry of the crystal. Namely,

$$H_B(\boldsymbol{r}+\boldsymbol{R}) = \left[\frac{1}{2m}\left(\left\{\boldsymbol{p}+e\,\underbrace{\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R})}_{\boldsymbol{A}(\boldsymbol{r})+\Delta\boldsymbol{A}}\right\}^2 + U(\boldsymbol{r})\right)\right] \neq H_B(\boldsymbol{r}) = \left[\frac{1}{2m}\left(\{\boldsymbol{p}+e\boldsymbol{A}(\boldsymbol{r})\}^2 + U(\boldsymbol{r})\right)\right]. \tag{5}$$

Here, $\Delta\boldsymbol{A}$ is given by

$$\Delta\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}+\boldsymbol{R}) - \boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}\begin{bmatrix} nbB \\ -maB \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} ma \\ nb \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} = \frac{1}{2}\left(\boldsymbol{R}\times\boldsymbol{B}\right), \tag{6}$$

and is evidently independent on $\boldsymbol{r}$.

B. Magnetic translation operators

Let us try to find an alternative set of eigenstates. Note here that, since the lattice symmetry for the electron must not be broken under the uniform magnetic field, we should have

$$H_B(\boldsymbol{r}+\boldsymbol{R})\,|\psi(\boldsymbol{r}+\boldsymbol{R})\rangle = E\,|\psi(\boldsymbol{r}+\boldsymbol{R})\rangle\,. \tag{7}$$

We then just need to put an extra phase factor on the conventional Bloch states to remove $\Delta\boldsymbol{A}$ in Eq. (6). In other words, we just need to perform some gauge-transformation on the conventional Bloch states. To this end, let us introduce the *magnetic translation operator* $T_{\boldsymbol{R}}$ given by

$$T_{\boldsymbol{R}} = \exp\left[i \underbrace{\frac{\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})}{\hbar}}_{\boldsymbol{q}} \cdot \boldsymbol{R}\right] = \exp\left[i\boldsymbol{q} \cdot \boldsymbol{R}\right], \tag{8}$$

where

$$\boldsymbol{q} = \frac{\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})}{\hbar} \tag{9}$$

is the *magnetic crystal momentum*. We have, for instance,

$$T_a \left|\psi(x,y)\right\rangle = e^{\frac{i}{\hbar}a\left(p_x + \frac{e}{2}yB\right)} \left|\psi(x,y)\right\rangle = e^{\frac{i}{\hbar}\left(\frac{1}{2}eBay\right)} \left|\psi(x+a,y)\right\rangle \tag{10}$$

$$T_b \left|\psi(x,y)\right\rangle = e^{\frac{i}{\hbar}b\left(p_x - \frac{e}{2}xB\right)} \left|\psi(x,y)\right\rangle = e^{-\frac{i}{\hbar}\left(\frac{1}{2}eBbx\right)} \left|\psi(x,y+b)\right\rangle. \tag{11}$$

With this $T_{\boldsymbol{R}}$, we find

$$\begin{aligned} H_B(\boldsymbol{r})\left(T_{\boldsymbol{R}}\left|\psi(\boldsymbol{r})\right\rangle\right) &= \exp\left[\frac{i}{\hbar}e\Delta\boldsymbol{A}\right] \underbrace{H_B(\boldsymbol{r}+\boldsymbol{R})\left|\psi(\boldsymbol{r}+\boldsymbol{R})\right\rangle}_{E\left|\psi(\boldsymbol{r}+\boldsymbol{R})\right\rangle} \\ &= E\left(T_{\boldsymbol{R}}\left|\psi(\boldsymbol{r})\right\rangle\right). \end{aligned} \tag{12}$$

This leads to

$$\left[H_B(\boldsymbol{r}), T_{\boldsymbol{R}}\right] = 0. \tag{13}$$

We thus should be able to find the simultaneous eigenstates of $H_B(\boldsymbol{r})$ and $T_{\boldsymbol{R}}$.

C. Magnetic Bloch states, magnetic unit cell, and magnetic Brillouin zone

The conventional translation operators defined by

$$T_{\boldsymbol{R}}^0 = \exp\left[i \underbrace{\frac{p}{\hbar}}_{\boldsymbol{k}} \cdot \boldsymbol{R}\right] = \exp\left[i\boldsymbol{k} \cdot \boldsymbol{R}\right] \tag{14}$$

are commutative. The simultaneous eigenstates of the system Hamiltonian and $T_{\boldsymbol{R}}^0$ constitute the Bloch states. On the other hand, the magnetic translation operators $T_{\boldsymbol{R}}$ along the different directions normally do not commute. For T_a and T_b defined by Eqs. (10) and (11), respectively, for instance, we have

$$T_bT_a = \exp\left[i \underbrace{\frac{e}{\hbar}}_{\Phi_0^{-1}} \underbrace{abB}_{\Phi}\right] T_aT_b = \exp\left[i\frac{\Phi}{\Phi_0}\right] T_aT_b. \tag{15}$$

This can be obtained by using the following formulas for operators $\hat{A}$ and $\hat{B}$:

$$e^{\hat{A}}e^{\hat{B}} = e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}e^{\hat{A}+\hat{B}}, \tag{16}$$

and

$$e^{\hat{B}}e^{\hat{A}} = e^{\frac{1}{2}\left[\hat{A},\hat{B}\right]}e^{\hat{A}+\hat{B}}, \tag{17}$$

which holds when the commutator $\left[\hat{A},\hat{B}\right]$ commutes with $\hat{A}$ and $\hat{B}$. Here, in our example,

$$\hat{A} = \frac{i}{\hbar}\left(p_x + \frac{e}{2}yB\right) = \frac{i}{\hbar}\left(-i\hbar\frac{\partial}{\partial x} + \frac{e}{2}yB\right) \tag{18}$$

$$\hat{B} = \frac{i}{\hbar}\left(p_y - \frac{e}{2}xB\right) = \frac{i}{\hbar}\left(-i\hbar\frac{\partial}{\partial y} - \frac{e}{2}xB\right), \tag{19}$$

and

$$\left[\hat{A},\hat{B}\right] = i\underbrace{\frac{e}{\hbar}}_{\Phi_0^{-1}}\underbrace{abB}_{\Phi}. \tag{20}$$

Here $\Phi = abB$ is the magnetic flux penetrating through the unit cell and $\Phi_0 = \frac{\hbar}{e}$ is flux quantum. The extra phase in Eq. (15) that leads to the non-commutativity of T_a and T_b is associated with the Aharonov-Bohm phase acquired by the electron traversing the edge of the unit cell.

Now let us assume that $\frac{\Phi}{\Phi_0} = \frac{p}{q}$, where p and q are coprime integers. Then, T_{qa} and T_b become commutative. The simultaneous eigenstates of $H_B(\boldsymbol{r})$, T_{qa}, and T_b constitute the eigenstates of the Schrödinger equation (2) for the n-th *magnetic Bloch band*. These states

$$\left|\psi_{n,\boldsymbol{q}}(\boldsymbol{r})\right\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\left|u_{n,\boldsymbol{q}}\right\rangle \tag{21}$$

are called the magnetic Bloch states. The resultant *magnetic unit cell* is q-times bigger in $x-$direction while the *magnetic Brillouin zone* is folded q-times.

III. LAUGHLIN'S THOUGHT EXPERIMENT [3, 5, 6]

Let us now consider the response of these 2D magnetic Bloch electrons when applying a bias voltage along $x-$direction: the situation in which the quantum Hall effect introduced in Sec. I was observed. The applied uniform electric field $\boldsymbol{E} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}$ leads to the linearly changing electrostatic potential $\Delta U(x,y) = Ex$, which again(!) breaks the translational symmetry of the crystal. To exploit the magnetic Bloch states, $\left|\psi_{n,\boldsymbol{q}}(\boldsymbol{r})\right\rangle$, in Eq. (21), let us perform Laughlin's ingenious device modification into a ring [5] as shown in Fig. 2. He considered the electric field $\boldsymbol{E}$ appeared through the electromotive force generated by the time-dependent flux penetrating the ring. The electric field $\boldsymbol{E}$ can then be given by the *uniform* but *time-dependent* vector potential $\boldsymbol{A}(t)$, namely,

$$\boldsymbol{E} = -\dot{\boldsymbol{A}}(t). \tag{22}$$

Then, under the electric field, the original Hamiltonian $H_B(\boldsymbol{r})$ in Eq. (2) is modified into

$$H_L(\boldsymbol{r},t) = \frac{1}{2m}\left(\boldsymbol{p} + e\left\{\boldsymbol{A}(\boldsymbol{r}) + \boldsymbol{A}(t)\right\}\right)^2 + U(\boldsymbol{r}). \tag{23}$$

The instantaneous eigenstates for $H_L(\boldsymbol{r},t)$ can also be given by the magnetic Bloch form

$$|\psi_{n,\tilde{\boldsymbol{q}}}(\boldsymbol{r})\rangle = e^{i\tilde{\boldsymbol{q}}\cdot\boldsymbol{r}}|u_{n,\tilde{\boldsymbol{q}}}(\boldsymbol{r})\rangle, \tag{24}$$

which is the same as the form given by Eq. (21) but $\boldsymbol{q}$ is replaced by

$$\tilde{\boldsymbol{q}} = \boldsymbol{q} + \frac{e}{\hbar}\boldsymbol{A}(t). \tag{25}$$

Since $\boldsymbol{A}(t)$ preserves the magnetic translational symmetry (i.e., $\left[\boldsymbol{A}(t), T_{\boldsymbol{R}}\right] = 0$) in any time, the magnetic crystal momentum $\boldsymbol{q}$ remains to be a good quantum number and satisfy

$$\dot{\boldsymbol{q}} = 0 \tag{26}$$

all the time. From Eq. (22) we thus have

$$\dot{\tilde{\boldsymbol{q}}} = \frac{e}{\hbar}\dot{\boldsymbol{A}}(t) = -\frac{e}{\hbar}\boldsymbol{E}. \tag{27}$$

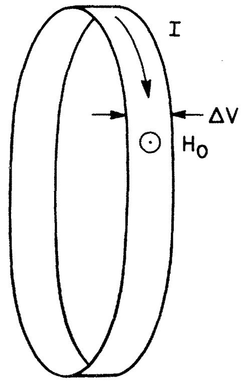

FIG. 2. Laughlin's modified sample geometry [5]. The ring constitutes the Hall bar, where the current I is applying along the ring while the Hall voltage ΔV is measured across the ring's edges under the out-of-plane magnetic field $\mu_0 H_0 = B$. The length of the ring is L. The current I is delivered through the electromotive force generated by the time-dependent flux penetrating the ring. The Hall resistance is $R_{\rm H} = \frac{\Delta V}{I}$.

Now, we are in a position to observe the response of the system shown in Fig. 2 upon an adiabatic variation of the magnetic flux penetrating the ring. By following the argument we used in discussing the adiabatic charge pumping, that is, *Thouless pumping* [7], let us calculate the Berry-curvature induced adiabatic current.

The first order correction to the adiabatic velocity $\boldsymbol{v}_{n,\tilde{\boldsymbol{q}}}(t)$ of the n-th magnetic Bloch band is given by the following Kubo formula:

$$
\begin{aligned}
\boldsymbol{v}_{n,\tilde{\boldsymbol{q}}}(t) &= -i \sum_{n'\neq n} \frac{1}{\epsilon_{n,\tilde{\boldsymbol{q}}} - \epsilon_{n',\tilde{\boldsymbol{q}}}} \left(\left\langle u_{n,\tilde{\boldsymbol{q}}} \middle| \frac{\partial H_L}{\partial \tilde{\boldsymbol{q}}} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial t} \right\rangle - \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial t} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial H_L}{\partial \tilde{\boldsymbol{q}}} \middle| u_{n,\tilde{\boldsymbol{q}}} \right\rangle \right) \\
&= -i \sum_{n'\neq n} \left(\left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial t} \right\rangle - \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial t} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \right\rangle \right). \qquad (28)
\end{aligned}
$$

Here, the eigen-energy $\epsilon_{n,\tilde{\boldsymbol{q}}}$ is given by

$$
\langle u_{n,\tilde{\boldsymbol{q}}} | e^{-i\tilde{\boldsymbol{q}}\cdot\boldsymbol{r}} H_L e^{i\tilde{\boldsymbol{q}}\cdot\boldsymbol{r}} | u_{n,\tilde{\boldsymbol{q}}} \rangle = \langle u_{n,\tilde{\boldsymbol{q}}} | \epsilon_{n,\tilde{\boldsymbol{q}}} | u_{n,\tilde{\boldsymbol{q}}} \rangle. \qquad (29)
$$

For deriving the quantized Hall resistance, we now use the relation

$$
\frac{\partial}{\partial t} = \dot{\tilde{\boldsymbol{q}}} \cdot \frac{\partial}{\partial \tilde{\boldsymbol{q}}} = -\frac{e}{\hbar} \boldsymbol{E} \cdot \frac{\partial}{\partial \tilde{\boldsymbol{q}}}, \qquad (30)
$$

which is obtained from Eq. (27). We then have

$$
\begin{aligned}
\boldsymbol{v}_{n,\tilde{\boldsymbol{q}}}(t) &= -i \sum_{n'\neq n} \left(\left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \left(-\frac{e}{\hbar} \underbrace{\boldsymbol{E}}_{\begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}} \cdot \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \right\rangle \right) - \left(-\frac{e}{\hbar} \underbrace{\boldsymbol{E}}_{\begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}} \cdot \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \right) \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{\boldsymbol{q}}} \right\rangle \right) \\
&= -i \sum_{n'\neq n} \left(\begin{bmatrix} \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \\ \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \\ 0 \end{bmatrix} \left(-\frac{e}{\hbar} E \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \right\rangle \right) - \left(-\frac{e}{\hbar} E \cdot \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \middle| u_{n',\tilde{\boldsymbol{q}}} \right\rangle \right) \begin{bmatrix} \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \right\rangle \\ \left\langle u_{n',\tilde{\boldsymbol{q}}} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y} \right\rangle \\ 0 \end{bmatrix} \right) \\
&= i \frac{e}{\hbar} E \begin{bmatrix} 0 \\ \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \right\rangle - \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y} \right\rangle \\ 0 \end{bmatrix} \qquad (31)
\end{aligned}
$$

IV. QUANTUM HALL EFFECT [4, 8]

A. Anomalous velocity

In an alternative mock-up expression, Eq. (31) can be written as

$$\boldsymbol{v}_{n,\tilde{\boldsymbol{q}}}(t) = \frac{e}{\hbar}\left(\boldsymbol{E}\times\boldsymbol{\Omega}_{n,\tilde{\boldsymbol{q}}}\right), \tag{32}$$

where

$$\boldsymbol{\Omega}_{n,\tilde{\boldsymbol{q}}} = \begin{bmatrix} 0 \\ 0 \\ i\left(\left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y}\right\rangle - \left\langle \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_y} \middle| \frac{\partial u_{n,\tilde{\boldsymbol{q}}}}{\partial \tilde{q}_x}\right\rangle\right)\end{bmatrix} \tag{33}$$

is nothing but the Berry curvature.

Putting the zero-order term, $\frac{\partial \epsilon_{n,\tilde{\boldsymbol{q}}}}{\hbar\partial\tilde{\boldsymbol{q}}}$, together, the adiabatic velocity of the electron in the n-th magnetic Bloch band with the magnetic crystal momentum $\tilde{\boldsymbol{q}}$ can be given by

$$\begin{aligned}\boldsymbol{v}^{\rm tot}_{n,\tilde{\boldsymbol{q}}} = \dot{\boldsymbol{r}}_{n,\tilde{\boldsymbol{q}}} &= \frac{\partial \epsilon_{n,\tilde{\boldsymbol{q}}}}{\hbar\partial\tilde{\boldsymbol{q}}} + \underbrace{\left\{-\frac{e}{\hbar}\left(\boldsymbol{E}\times\Omega_{n,\boldsymbol{k}}\right)\right\}}_{\boldsymbol{v}_{n,\boldsymbol{k}}} \\ &= \frac{\partial \epsilon_{n,\tilde{\boldsymbol{q}}}}{\hbar\partial\tilde{\boldsymbol{q}}} + \dot{\tilde{\boldsymbol{q}}}\times\Omega_{n,\boldsymbol{k}}. \end{aligned} \tag{34}$$

The first order correction $\boldsymbol{v}_{n,\boldsymbol{k}}$ is indeed transverse to the electric field $\boldsymbol{E}$ and is called *anomalous velocity*, produces the dissipationless current, and is responsible for the *quantum Hall effect* as we shall see in the following.

B. Quantum Hall effect

The Hall current density $j_{\rm H}$ perpendicular to $\boldsymbol{E}$, which results from the second term in the velocity Eq. (34), can be expressed as

$$j_{\rm H} = -e\sum_n \int_{\rm MBZ} \frac{d\tilde{\boldsymbol{q}}}{(2\pi)^2}\boldsymbol{v}_{n,\tilde{\boldsymbol{q}}} \tag{35}$$

$$= E\underbrace{\frac{e^2}{\hbar}\sum_n \int_{\rm MBZ} \frac{d\tilde{\boldsymbol{q}}}{(2\pi)^2}\Omega_{n,\tilde{\boldsymbol{q}}}}_{\sigma_{xy}} \tag{36}$$

where the integration is over the magnetic Brillouin zone (MBZ) [3, 4, 8]. Since $\sum_n \int_{\rm MBZ}\frac{d\tilde{\boldsymbol{q}}}{2\pi}\Omega_{n,\tilde{\boldsymbol{q}}}$ can have some integer value Z [3, 4, 8] (this follows from the similar argument employed when discussing Thouless pumping [7]), the Hall conductivity σ_{xy} in Eq. (36) can be written by the simple form

$$\sigma_{xy} = \frac{e^2}{h}Z, \tag{37}$$

and seen to be *quantized* in units of $\frac{e^2}{h}$! In our current setting, the Hall resistance can be expressed as

$$R_{\rm H} = \frac{EL}{j_{\rm H}L} = \frac{1}{\sigma_{xy}}, \tag{38}$$

where L is the length of the ring used in our thought experiment. Thus, we arrive at the conclusion that the observed Hall resistance indicated in Eq. (1) can be interpreted as the result of this quantized Hall conductivity with $Z = 4$!

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