Clash course of superconductivity

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We review the *phenomenological* theory of superconductivity.

I. FROM SCHRÖDINGER TO LONDON [1]

The *Schrödinger equation* of a charged particle in an electromagnetic field with a vector potential $\boldsymbol{A}$ as well as a scalar potential ϕ can be given by

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) &= \mathcal{H}\psi(\boldsymbol{r},t) \\ &= \frac{1}{2m}\left(-i\hbar\nabla - q\boldsymbol{A}\right)\left(-i\hbar\nabla - q\boldsymbol{A}\right)\psi + q\phi\psi \end{aligned} \tag{1}$$

Since the probability density $P(\boldsymbol{r},t)$ in quantum mechanics is given in terms of the wave function $\psi(\boldsymbol{r},t)$ by

$$P(\boldsymbol{r},t) = \psi^*(\boldsymbol{r},t)\psi(\boldsymbol{r},t), \tag{2}$$

the *probability current* $\boldsymbol{J}(\boldsymbol{r},t)$ can be obtained by

$$\frac{\partial}{\partial t}P(\boldsymbol{r},t) = -\nabla\cdot\boldsymbol{J}(\boldsymbol{r},t), \tag{3}$$

which leads to

$$\boldsymbol{J}(\boldsymbol{r},t) = \frac{1}{2}\left(\left(\frac{-i\hbar\nabla - q\boldsymbol{A}}{m}\psi\right)^*\psi + \psi^*\left(\frac{-i\hbar\nabla - q\boldsymbol{A}}{m}\psi\right)\right). \tag{4}$$

Here is the crucial point: we new consider the wave function $\psi(\boldsymbol{r})$ in Eq. (1) as a *macroscopic* one by identifying it as an *order parameter* of a superconducting metal;

$$\psi(\boldsymbol{r},t) = \sqrt{\rho(\boldsymbol{r})}e^{i\theta(\boldsymbol{r})}, \tag{5}$$

where ρ is the *charge density* and θ is the *phase*. Equation. (4) is now nothing but the *electric current* as opposed to the probability current. The current can be explicitly given in terms of ρ and θ by

$$\boldsymbol{J}(\boldsymbol{r}) = \frac{\hbar}{m}\left(\nabla\theta(\boldsymbol{r}) - \frac{q}{\hbar}\boldsymbol{A}(\boldsymbol{r})\right)\rho(\boldsymbol{r}). \tag{6}$$

Without the first term Eq. (6) is the so-called *second London equation* [2] [4];

$$\boldsymbol{J}(\boldsymbol{r}) = -\frac{q}{m}\rho(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}), \tag{7}$$

which explains the *perfect conductivity* as well as the *Meissner effect* as follows.

II. CONSEQUENCES

A. Perfect conductivity [2]

Taking time derivative of Eq. (7), we have

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = -\frac{q\rho(\boldsymbol{r})}{m}\dot{\boldsymbol{A}}(\boldsymbol{r}). \tag{8}$$

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Since $\dot{\boldsymbol{A}}(\boldsymbol{r}) = -\boldsymbol{E}(\boldsymbol{r})$, we have

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = \frac{q\rho(\boldsymbol{r})}{m}\boldsymbol{E}(\boldsymbol{r}), \tag{9}$$

which suggests the perfect conductivity. The situation is analogous to the lossless free mass system:

$$\underbrace{\dot{\boldsymbol{p}}}_{m\dot{\boldsymbol{J}}(\boldsymbol{r})} = \underbrace{\boldsymbol{F}}_{\rho(\boldsymbol{r})q\boldsymbol{E}}. \tag{10}$$

There were, on the other hand, a loss term, we have

$$\dot{\boldsymbol{p}} = -\frac{1}{\tau}\boldsymbol{p} + \boldsymbol{F}, \tag{11}$$

and thus

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = -\frac{1}{\tau}\boldsymbol{J}(\boldsymbol{r}) + \frac{\rho(\boldsymbol{r})q}{m}\boldsymbol{E}, \tag{12}$$

where τ is the *relaxation time* [3]. Suppse that the τ is very short, we can set $\dot{\boldsymbol{J}}(\boldsymbol{r}) = 0$ in Eq. (12). This gives us

$$\boldsymbol{J}(\boldsymbol{r}) = \underbrace{\frac{\rho(\boldsymbol{r})q}{m}\tau}_{\sigma}\boldsymbol{E}, \tag{13}$$

and recovering the standard *Ohm's law* within the Drude theory [3].

B. Meissner effect [1]

Let us now analyze the consequence of the London equation (6) upon the magnetic field in a superconductor. Let us start by writing the Maxwell equation

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}(\boldsymbol{r}) \tag{14}$$

in terms of the vecotor potential $\boldsymbol{A}(\boldsymbol{r})$, that is

$$\nabla \times \underbrace{(\nabla \times \boldsymbol{A}(\boldsymbol{r}))}_{\boldsymbol{B}(\boldsymbol{r})} = \mu_0 \boldsymbol{J}(\boldsymbol{r}). \tag{15}$$

With the Coulomb gauge $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$ and using vector identity

$$\nabla \times \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \nabla\left(\nabla \cdot \boldsymbol{A}(\boldsymbol{r})\right) - \nabla^2 \boldsymbol{A}(\boldsymbol{r}) \tag{16}$$

Equation 15 becomes

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\mu_0 \boldsymbol{J}(\boldsymbol{r}). \tag{17}$$

Using the London equation (6) we have a simple equation for $\boldsymbol{A}(\boldsymbol{r})$:

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\underbrace{\frac{\mu_0 \rho(\boldsymbol{r})q}{m}}_{\frac{1}{\lambda_L^2}}\boldsymbol{A}(\boldsymbol{r}), \tag{18}$$

where

$$\lambda_L = \sqrt{\frac{m}{\mu_0 \rho(\boldsymbol{r})q}} \tag{19}$$

is called the *London penetration depth* [2, 3]. The solution of Eq. (18) is

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}_0 \exp\left(-\frac{\boldsymbol{r}}{\lambda_L}\right). \tag{20}$$

The same conclusion can be delivered for the magnetic field $\boldsymbol{B}(\boldsymbol{r})$, that is

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_0 \exp\left(-\frac{\boldsymbol{r}}{\lambda_L}\right), \tag{21}$$

suggesting the exponential decay of the magnetic field inside the superconductor, i.e., the *Meissner effect*.

C. Flux quantization [1]

The existence of the θ term in Eq. (6) produces an even more remarkable phenomenon, *flux quantization*. Let us consider a superconducting ring under the magnetic field. Since the superconducting current flows only near the surface down to the London penetration depth, Eq. (19), the interior current of the ring should be zero. From Eq. (6) this situation leads to

$$\hbar \nabla \theta(\boldsymbol{r}) = q\boldsymbol{A}(\boldsymbol{r}). \tag{22}$$

Taking the line integral along the interior of the ring, we have, from the single-valuedness of the wave function,

$$\hbar \underbrace{\oint \nabla\theta(\boldsymbol{r}) \cdot d\boldsymbol{s}}_{2\pi n} = q \underbrace{\oint \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{s}}_{\Phi}, \tag{23}$$

and reach the conclusion that the flux Φ has to be *quantized* as

$$\Phi = \frac{2\pi\hbar}{q} n \tag{24}$$

with n being any integers (0,1,2, $\cdots$) and q turning out to be $2e$ reflecting the fact that the electrons pairing up as the *Cooper pairs* and being condensed in the ground state (BCS state) in the superconducting metals.

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[1] R. P. Feynman, R. B. Leighton and M. Sands, *Feynman Lectures on Physics*, vol. III, (Basic book, New York, 2011).
[2] A. Altland and B. D. Simons, *Condensed Matter Field Theory* (Cambridge University Press; 2nd edition, Cambridge, 2010).
[3] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Brooks/Cole, Belmont, 1976).
[4] The *first London equation* is $\left(\frac{q\rho(\boldsymbol{r})}{m} - \nabla^2\right)\boldsymbol{B} = 0$. See Ref [2] for details.