Solid State Physics IV
-Part II : Macroscopic Quantum Phenomena

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We study an open quantum system in which a (macroscopic) harmonic oscillator is coupled to an environment that is modeled as a continuum boson field. We shall learn that the quantum noise spectrum of a resister (Ohmic noise) is asymmetric in frequency because of the non-commutativity of the Boson field operators and then the *quantum dissipation-fluctuation theorem* is presented.

II. DAMPED HARMONIC OSCILLATORS

A. Quantization of environment

We will start with treating an energy-dissipative electronic resister as an infinite set of energy-conservative harmonic oscillators, i.e., a Boson field, which plays a role of an environment for a concerned quantum system (in particular an LC circuit). This analysis serves as a model for more general *Ohmic (linear dissipative) environments.*

1. Note on boundary conditions

There are many confusing points we have to be careful in quantizing Boson fields. Among other things I would like to emphasize in particular two points, the issue of dimension, which is discussed a little bit in the last part of Sec.II A 2, and the issue of boundary condition, which we shall look at here by summarizing major boundary conditions used in the literature to help clarify the differences:

Periodic (Born-von Karman) boundary condition (Traveling wave [*complex*])

- Boundary condition: $\varphi(0) = \varphi(L)$
- Variable: $\varphi(x) = \frac{1}{\sqrt{L}} \sum_{k_n} \varphi_n e^{ik_n x}$
- Fourier transform: $\varphi_n = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) e^{-ik_n x}$
- $k_n = \frac{2\pi}{L} n$
- $n = \underbrace{0, \pm 1, \pm 2, \cdots, \pm \frac{N}{2} - 1, \frac{N}{2}}_{\text{N points}}$

Dirichlet boundary condition (Fixed-end standing wave [*real*])

- Boundary condition: $\varphi(0) = \varphi(L) = 0$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum_{k_n \geq 0} \varphi_n^{(s)} \sin(k_n x)$
- Fourier transform: $\varphi_n^{(s)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \sin(k_n x)$
- $k_n = \frac{\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, N-1, N}_{\text{N points}}$

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Neumann boundary condition (Open-end standing wave [*real*])

- Boundary condition: $\frac{\partial}{\partial x}\varphi(0) = \frac{\partial}{\partial x}\varphi(L) = 0$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum\limits_{k_n \geq 0} \varphi_n^{(c)} \cos(k_n x)$
- Fourier transform: $\varphi_n^{(c)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \cos(k_n x)$
- $k_n = \frac{\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, N-1, N}_{\text{N points}}$

Mixed boundary condition (Cosine and sine modes [*real*])

- Boundary condition: $\varphi(0) = \varphi(L)$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum\limits_{k_n \geq 0} \left(\varphi_n^{(c)} \cos(k_n x) - \varphi_n^{(s)} \sin(k_n x) \right)$
- Fourier transform: $\begin{array}{l} \varphi_n^{(c)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \cos(k_n x) \\ \varphi_n^{(s)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \sin(k_n x) \end{array}$
- $k_n = \frac{2\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, \frac{N}{2} - 1}_{\text{N/2 points}}$

We will henceforth chiefly employ periodic (Born-von Karman) boundary condition for quantizing Boson field.

2. Transmission line [1]

The language we have developed for treating an atomic chain as a Boson field can be translated into that for a coaxial transmission line (see the correspondences table, Table II A 2). The field equation of the transmission line can be read as

$$\left(\frac{1}{v_p^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) \varphi(x, t) = 0, \tag{1}$$

where

$$v_p = \frac{1}{\sqrt{lc}} \tag{2}$$

TABLE I: Atomic chain - Transmission line - EM traveling mode

	Atomic chain	Transmission line	EM traveling mode
Mass density	ρ	c	ϵ_0
Elastic const.	c_{11}	$\frac{1}{l}$	$\frac{1}{\mu_0}$
Velocity	$v_s = \sqrt{\frac{c_{11}}{\rho}}$	$v_p = \frac{1}{\sqrt{cl}}$	$c_v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$
Impedance	$Z_s = \sqrt{\frac{1}{\rho c_{11}}}$	$Z_p = \sqrt{\frac{l}{c}}$	$Z_v = \sqrt{\frac{\mu_0}{\epsilon_0}}$
Displacement	$q(x, t)$	$\varphi(x, t)$	$A(\omega)$
Momentum	$p(x, t)$	$q(x, t)$	$\Pi(\omega)$

with l being the inductance per unit length and c being the capacitance to the ground per unit length. The *flux variable* $\varphi(x,t)$ [2] is related to the *local voltage* $V(x,t)$ as

$$\varphi(x,t) = \int_{-\infty}^{t} d\tau V(x,\tau), \tag{3}$$

and thus

$$V(x,t) = \dot{\varphi}(x,t). \tag{4}$$

The *local current* $I(x,t)$, on the other hand, is given by

$$I(x,t) = -\frac{\left(\frac{\partial\varphi(x,t)}{\partial x}\right) dx}{\left(\frac{\partial L}{\partial x}\right) dx} = -\frac{1}{l}\frac{\partial}{\partial x}\varphi(x,t). \tag{5}$$

With these relations the Lagrangian density can be deduced as

$$\begin{aligned} \mathcal{L}(x,t) &= \underbrace{\frac{c}{2}V(x,t)^2}_{\text{Kinetic energy}} - \underbrace{\frac{l}{2}I(x,t)^2}_{\text{Potential energy}} \\ &= \frac{c}{2}\left(\dot{\varphi}(x,t)\right)^2 - \frac{l}{2}\left(-\frac{1}{l}\frac{\partial}{\partial x}\varphi(x,t)\right)^2 \\ &= \frac{c}{2}\left(\dot{\varphi}(x,t)\right)^2 - \frac{1}{2l}\left(\frac{\partial}{\partial x}\varphi(x,t)\right)^2, \end{aligned} \tag{6}$$

from which the Euler-Lagrange equation can be deduced as

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} + \underbrace{\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\frac{\partial}{\partial x}\varphi}}_{\text{extra term}} = 0, \tag{7}$$

which is exactly the wave equation we had in Eq. (1).

The momentum conjugate to $\varphi(x,t)$ is indeed charge density,

$$q(x,t) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = cV(x,t). \tag{8}$$

For a typical coaxial cable (RG-58U)

$$l = -\frac{\frac{\partial}{\partial x}\varphi(x,t)}{I(x,t)} = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \sim 200 \text{ nH/m} \tag{9}$$

$$c = \frac{q(x,t)}{V(x,t)} = \frac{2\pi\epsilon}{\ln\left(\frac{b}{a}\right)} \sim 80 \text{ pF/m}, \tag{10}$$

which lead to

$$v_p = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu_0\epsilon}} \sim 2.5\times 10^8 \text{ m/s} \tag{11}$$

$$Z_p = \sqrt{\frac{l}{c}} = \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon}}\ln\left(\frac{b}{a}\right) \sim 50\ \Omega, \tag{12}$$

where a and b are the radii of the center conductor and the outer conductor, respectively.

The Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2c}q(x,t)^2 + \frac{1}{2l}\left(\frac{\partial}{\partial x}\varphi(x,t)\right)^2 \tag{13}$$

and the Hamiltonian is obtained by integrating the Hamiltonian density Eq. (13) over the length L:

$$H = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \mathcal{H}(x). \tag{14}$$

In reciprocal space the flux and charge variables are defined with the aforementioned periodic boundary condition as

$$\begin{aligned} \varphi_n &= \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) e^{-ik_n x} && (15) \\ q_n &= \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx q(x) e^{ik_n x}, && (16) \end{aligned}$$

with the commutation relation

$$[\varphi_n, q_{n'}] = i\hbar \delta_{nn'} \tag{17}$$

and the Hamiltonian can be rewritten in terms of φ_n and q_n as

$$H = \sum_{k_n} \mathcal{H}_n \tag{18}$$

with

$$\begin{aligned} \mathcal{H}_n &= \frac{1}{2c} q_n q_{-n} + \frac{1}{2l} \varphi_n \varphi_{-n} \\ &= \hbar\omega_n \left(\hat{c}_n^\dagger \hat{c}_n + \frac{1}{2} \right), && (19) \end{aligned}$$

where the annihilation and creation operators for the Boson field are respectively defined by

$$\begin{aligned} \hat{c}_n &= \sqrt{\frac{c\omega_n}{2\hbar}} \left(\varphi_{-n} + \frac{i}{c\omega_n} q_n \right) && (20) \\ \hat{c}_n^\dagger &= \sqrt{\frac{c\omega_n}{2\hbar}} \left(\varphi_n - \frac{i}{c\omega_n} q_{-n} \right), && (21) \end{aligned}$$

with the commutation relation

$$[\hat{c}_n, \hat{c}_{n'}] = \delta_{nn'} \tag{22}$$

These are the same as the forms for the φ-representation of a LC circuit except for the index n and that the quantities φ_n and q_n are complex variable due to the fact that the periodic boundary condition is used to formulate.

Taking the *second* continuum limit $L \to \infty$ makes the sum on k_n in Eq. (18) changed into the integral over k:

$$H = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar\omega_k \left(\hat{c}^\dagger(k) \hat{c}(k) + \frac{1}{2} \right), \tag{23}$$

where

$$\begin{aligned} \hat{c}(k) &= \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(-k) + \frac{i}{c\omega_k} q(k) \right) && (24) \\ \hat{c}^\dagger(k) &= \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(k) - \frac{i}{c\omega_k} q(-k) \right), && (25) \end{aligned}$$

with

$$\begin{aligned} \varphi(k) &= \lim_{L\to\infty} \sqrt{L} \varphi_n = \int_{-\infty}^{\infty} dx \varphi(x) e^{-ikx} && (26) \\ q(k) &= \lim_{L\to\infty} \sqrt{L} q_n = \int_{-\infty}^{\infty} dx q(x) e^{ikx}. && (27) \end{aligned}$$

The commutation relation for the canonical operators is

$$
\begin{aligned}
[\varphi(k), q(k')] &= \left[\int_{-\infty}^{\infty} dx \varphi(x) e^{-ikx}, \int_{-\infty}^{\infty} dx' q(x') e^{ik'x'}\right] \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \underbrace{[\varphi(x), q(x')]}_{i\hbar\delta(x-x')} e^{-i\left(kx-k'x'\right)} \\
&= i\hbar \underbrace{\int_{-\infty}^{\infty} dx e^{-i\left(k-k'\right)x}}_{2\pi\delta(k-k')} \\
&= i\hbar\, 2\pi\delta(k-k'), \qquad (28)
\end{aligned}
$$

and thus that for the annihilation and creation operators is

$$[\hat{c}(k), \hat{c}^{\dagger}(k)] = 2\pi\delta(k-k') \qquad (29)$$

from Eqs. (24), (25), and (28).

We have to be aware that the dimensions of the operators change when the continuum limits are taken. At the *first* continuum limit we have

$$\varphi(x) = \lim_{\substack{a\to 0 \\ N_a\to\infty}} \frac{\varphi_n(t)}{\sqrt{a}} \qquad (30)$$

$$q(x) = \lim_{\substack{a\to 0 \\ N_a\to\infty}} \frac{q_n(t)}{\sqrt{a}}, \qquad (31)$$

thus the dimension of $\varphi(x)$ and $q(x)$ differs from that of φ_n and q_n by factor of $[\frac{1}{\sqrt{\text{length}}}]$. At the *second* continuum limit, on the other hand, we have

$$\varphi(k) = \lim_{L\to\infty} \sqrt{L}\varphi_n \qquad (32)$$

$$q(k) = \lim_{L\to\infty} \sqrt{L} q_n, \qquad (33)$$

thus the dimension of $\varphi(k)$ and $q(k)$ differs from that of φ_n and q_n by factor of $[\sqrt{\text{length}}]$. This difference leads to the different commutators in Eqs (17) and (28), which in turn leads to the different commutators in Eqs. (22) and (29) for the annihilation and creation operators.

3. Johnson-Nyquist noise

The Heisenberg equations of motion for $\hat{c}(k)$ and $\hat{c}^{\dagger}(k)$ are

$$\dot{\hat{c}}(k,t) = \frac{i}{\hbar}\left[H, \hat{c}(k,t)\right] = -iv_p k\hat{c}(k,t) = -i\omega_k \hat{c}(k,t) \qquad (34)$$

$$\dot{\hat{c}}^{\dagger}(k,t) = \frac{i}{\hbar}\left[H, \hat{c}^{\dagger}(k,t)\right] = iv_p k\hat{c}^{\dagger}(k,t) = i\omega_k \hat{c}^{\dagger}(k,t) \qquad (35)$$

thus we have the plane wave solutions:

$$\hat{c}(k,t) = \hat{c}(k,0)e^{-i\omega_k t} \qquad (36)$$

$$\hat{c}^{\dagger}(k,t) = \hat{c}^{\dagger}(k,0)e^{i\omega_k t}. \qquad (37)$$

With these results the charge variable $q(x,t)$ is given by

$$
\begin{aligned}
q(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} q(k,t) e^{ikx} \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} i\sqrt{\frac{\hbar\omega_k c}{2}} \left(\hat{c}^{\dagger}(-k,t) - \hat{c}(k,t)\right) e^{ikx} \\
&= -i\int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar\omega_k c}{2}} \left(\hat{c}(k,0)e^{i(kx-\omega_k t)} - h.c.\right), \qquad (38)
\end{aligned}
$$

which is indeed manifestly real as it has to be. The voltage V(x,t), which is also a real quantity, can be written in terms of $q(x,t)$ as

$$V(x,t) = \frac{q(x,t)}{c} = -i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sqrt{\frac{\hbar\omega_k}{2c}}\left(\hat{c}(k,0)e^{i(kx-\omega_k t)} - h.c.\right). \tag{39}$$

We now identify the modes with positive k as the right-moving modes and those with negative k as the left-moving modes. The right-moving voltage $V^{\rightarrow}(x,t)$ can thus be given by

$$\begin{aligned} V^{\rightarrow}(x,t) &= -i\int_{0}^{\infty}\frac{dk}{2\pi}\sqrt{\frac{\hbar\omega_k}{2c}}\left(\hat{c}(k,0)e^{i(kx-\omega_k t)} - h.c.\right) \\ &= -i\int_{0}^{\infty}\frac{v_p dk}{2\pi}\sqrt{\frac{\hbar\omega_k}{2cv_p}}\left(\frac{\hat{c}(k,0)}{\sqrt{v_p}}e^{i(kx-\omega_k t)} - h.c.\right) \\ &= -i\int_{0}^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(kx-\omega t)} - h.c.\right) \end{aligned} \tag{40}$$

and the left-moving voltage can similarly given by

$$V^{\leftarrow}(x,t) = -i\int_{-\infty}^{0}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(kx-\omega t)} - h.c.\right), \tag{41}$$

where $\hat{c}(\omega) = \frac{\hat{c}(k,0)}{\sqrt{v_p}}$, which satisfies the commutation relation:

$$[\hat{c}(\omega), \hat{c}^{\dagger}(\omega')] = [\frac{\hat{c}(k)}{\sqrt{v_p}}, \frac{\hat{c}^{\dagger}(k')}{\sqrt{v_p}}] = \frac{2\pi}{v_p}\delta(k-k') = 2\pi\delta(\omega-\omega'). \tag{42}$$

While the average voltage fluctuation $\langle V(x,t)\rangle_t$ is zero under the thermal equilibrium the variance is not, which is basically the *Johnson-Nyquist noise*. By evaluating the variance, or rather the spectral density $S_{VV}(\omega)$, we shall find the quantum version of the *Nyquist formula*. Let us consider the auto-correlation of the voltage at the open terminal at $x = 0$ of a semi-infinite transmission line with the characteristic impedance $Z_p = \sqrt{\frac{l}{c}}$, which can be given by

$$\begin{aligned} \langle V(0,t+\tau)V(0,t)\rangle_t &= \langle (V^{\rightarrow}(0,t+\tau) + V^{\leftarrow}(0,t+\tau))\,(V^{\rightarrow}(0,t) + V^{\leftarrow}(0,t))\rangle_t \\ &= 4\langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle_t, \end{aligned} \tag{43}$$

where the stationarity leads to the first equation, and $V(x,t) = V^{\rightarrow}(x,t) + V^{\leftarrow}(x,t)$ and $V^{\rightarrow}(x,t) = V^{\leftarrow}(x,t)$ for the open terminal lead to the second and third equation, respectively.

For the situation in which the stationarity condition is satisfied the spectral density is obtained via the *Wiener-Khinchin theorem*:

$$\begin{aligned} S_{VV}(\Omega) &= \int_{-\infty}^{\infty} d\tau \langle V(0,t+\tau)V(0,t)\rangle_t e^{i\Omega\tau} \\ &= 4\int_{-\infty}^{\infty} d\tau \langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle_t e^{i\Omega\tau} = 4S^{\rightarrow}_{VV}(\Omega). \end{aligned} \tag{44}$$

With Eq. (40) we have

$$
\begin{aligned}
S^{\rightarrow}_{VV}(\Omega) &= \int_{-\infty}^{\infty} d\tau \langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle e^{i\Omega\tau} \\
&= -\int_{-\infty}^{\infty} d\tau \int_0^{\infty}\frac{d\omega'}{2\pi}\int_0^{\infty}\frac{d\omega'}{2\pi}\frac{\hbar Z_p}{2}\sqrt{\omega\omega'}\left(\underbrace{\langle \hat{c}(\omega)c(\omega')e^{-i(\omega+\omega')t}\rangle}_{0} - \underbrace{\langle \hat{c}(\omega)c^{\dagger}(\omega')e^{-i(\omega-\omega')t}\rangle}_{(n(\omega)+1)2\pi\delta(\omega-\omega')}\right)e^{i(\Omega-\omega)\tau} \\
&\qquad + \left(-\underbrace{\langle \hat{c}^{\dagger}(\omega)c(\omega')e^{-i(-\omega+\omega')t}\rangle}_{n(\omega)2\pi\delta(\omega-\omega')} + \underbrace{\langle \hat{c}^{\dagger}(\omega)c^{\dagger}(\omega')e^{-i(-\omega-\omega')t}\rangle}_{0}\right)e^{i(\Omega+\omega)\tau} \\
&= \int_{-\infty}^{\infty} d\tau \int_0^{\infty}\frac{d\omega}{2\pi}\frac{\hbar\omega Z_p}{2}\left((n(\omega)+1)\,e^{i(\Omega-\omega)\tau} + n(\omega)e^{i(\Omega+\omega)\tau}\right) \\
&= \int_0^{\infty} d\omega \frac{\hbar\omega Z_p}{2}\left((n(\omega)+1)\,\delta(\Omega-\omega) + n(\omega)\delta(\Omega+\omega)\right) \\
&= \frac{\hbar|\Omega|Z_p}{2}\left((n(\Omega)+1)\,\Theta(\Omega) + n(|\Omega|)\Theta(-\Omega)\right),
\end{aligned} \tag{45}
$$

where $\Theta(x)$ is the step function. Thus we have the voltage noise spectrum:

$$
S_{VV}(\Omega) = 4S^{\rightarrow}_{VV}(\Omega) = 2\hbar|\Omega|Z_p\left((n(\Omega)+1)\,\Theta(\Omega) + n(|\Omega|)\Theta(-\Omega)\right). \tag{46}
$$

Let us here take a step back and see what is going on here. For the real-valued *classical* variable $V(\tau)$ its auto-correlation function $G_{VV}(\tau) = \langle V(\tau)V(0)\rangle$ is also real. The commutativity of classical variable also suggests $G_{VV}(\tau) = G_{VV}(-\tau)$, that is, the auto-correlation is symmetric in time. This leads to the *symmetric-in-frequency* power spectrum:

$$
\begin{aligned}
S_{VV}(-\Omega) &= \int_{-\infty}^{\infty} d\tau G_{VV}(\tau)e^{-i\Omega\tau} \\
&= \int_{\infty}^{-\infty} (-d\tau)\underbrace{G_{VV}(-\tau)}_{G_{VV}(\tau)}e^{i\Omega\tau} = S_{VV}(\Omega).
\end{aligned} \tag{47}
$$

For the real-valued *quantum* variable $V(\tau)$, however, its auto-correlation function $G_{VV}(\tau)$ is not necessarily real! Let us see this in the following simple argument with a LC circuit. The real-valued flux variable is given by

$$
\varphi(t) = \sqrt{\frac{\hbar}{2C_0\omega}}\left(\hat{c}(t)+\hat{c}^{\dagger}(t)\right) = \sqrt{\frac{\hbar Z_0}{2}}\left(\hat{c}e^{-i\omega_0 t}+\hat{c}^{\dagger}e^{i\omega_0 t}\right), \tag{48}
$$

which is manifestly hermitian. The auto-correlation function is, however, *not* hermitian:

$$
\begin{aligned}
G_{\varphi\varphi}(\tau) &= \frac{\hbar Z_0}{2}\left(\langle \hat{c}\hat{c}^{\dagger}\rangle e^{-i\omega_0 t} + \langle \hat{c}^{\dagger}\hat{c}\rangle e^{i\omega_0 t}\right) \\
&= \frac{\hbar Z_0}{2}\left((n(\omega_0)+1)e^{-i\omega_0 t} + n(\omega_0)e^{i\omega_0 t}\right).
\end{aligned} \tag{49}
$$

Thus we arrive the *asymmetric-in-frequency* power spectrum power spectrum:

$$
S_{\varphi\varphi}(\Omega) = \int_{-\infty}^{\infty} d\tau G_{\varphi\varphi}(\tau)e^{i\Omega\tau} = \frac{\hbar Z_0}{2}\left((n(\omega_0)+1)2\pi\delta(\Omega-\omega_0) + n(\omega_0)2\pi\delta(\Omega+\omega_0)\right). \tag{50}
$$

Since $V(t) = \dot{\varphi}(t)$ by the similar argument we have the *asymmetric-in-frequency* power spectrum:

$$
S_{VV}(\Omega) = \frac{\hbar\omega_0^2 Z_0}{2}\left((n(\omega_0)+1)2\pi\delta(\Omega-\omega_0) + n(\omega_0)2\pi\delta(\Omega+\omega_0)\right), \tag{51}
$$

which is the discrete version of the spectral density in Eq. (46). We see that the non-commutativity of the quantum operators $\varphi(t)$ and $V(t)$ with those in different time is the culprit of the *asymmetric-in-frequency* power spectrum power spectrum. We also see that the result we have in Eq. (46) can be obtained by adding the contribution of infinitely many LC circuits with different frequencies.

This expression Eq. (46) can recast into more compact form:

$$
\begin{aligned}
S_{VV}(\Omega) &= 2Z_p\left(\hbar\Omega\left(\frac{1}{e^{\frac{\hbar\Omega}{k_BT}}-1}+1\right)\Theta(\Omega)-\hbar\Omega\left(\frac{1}{e^{-\frac{\hbar\Omega}{k_BT}}-1}\right)\Theta(-\Omega)\right)\\
&= 2Z_p\left(\hbar\Omega\left(\frac{1}{1-e^{-\frac{\hbar\Omega}{k_BT}}}\right)\Theta(\Omega)+\hbar\Omega\left(\frac{1}{1-e^{-\frac{\hbar\Omega}{k_BT}}}\right)\Theta(-\Omega)\right)\\
&= \left(\frac{2Z_p\hbar\Omega}{1-e^{-\frac{\hbar\Omega}{k_BT}}}\right). \qquad (52)
\end{aligned}
$$

This is called a double-sided spectral density where the frequency Ω runs from negative to positive. The single-sided spectral density is, on the other hand, given by

$$
\begin{aligned}
\bar{S}_{VV}(\Omega) = S_{VV}(\Omega)+S_{VV}(-\Omega) &= \left(\frac{2Z_p\hbar\Omega}{1-e^{-\frac{\hbar\Omega}{k_BT}}}\right)+\left(\frac{-2Z_p\hbar\Omega}{1-e^{\frac{\hbar\Omega}{k_BT}}}\right)\\
&= 2Z_p\hbar\Omega\coth(\frac{\hbar\Omega}{2k_BT}) \qquad (53)\\
&= 4Z_p\hbar\Omega\left\{\underbrace{\frac{1}{e^{\frac{\hbar\Omega}{k_BT}}-1}}_{n(\hbar\Omega)}+\frac{1}{2}\right\}, \qquad (54)
\end{aligned}
$$

where the frequency Ω runs only in the positive direction. In the last line we can recognize the contribution of the *zero point fluctuation*, $2Z_p\hbar\Omega$, to the noise spectral density explicitly. Equation (54) is called *quantum dissipation-fluctuation theorem*, which connects the apparently unrelated two quantities; the transport coefficient Z_p and the noise spectral density $\bar{S}_{VV}(\Omega)$.

By taking the classical limit $k_{BT} \gg \hbar\Omega$ the spectral density Eq. (54) becomes

$$
\bar{S}_{VV}(\Omega) = 4Z_pk_BT, \qquad (55)
$$

which is the well-known *Johnson-Nyquist formula*, where the spectrum is proportional to the impedance Z_p and temperature T

4. Ohmic environment

We are thus able to treat a dissipative element characterized by the impedance Z_p quantum mechanically. The quantum dissipation fluctuation theorem Eq. (54) shows peculiar quantum effect which manifest itself as the *asymmetric-in-frequency* power spectrum in the quantum regime $k_{BT} \leq \hbar\Omega$. That the dissipative elements can be treated as a collection of conservative (reactive) elements is essentially the way in which the *Caldeira-Leggett model* deals with resisters quantum mechanically [1, 2]. The environment which is characterized by the *frequency-independent* impedance Z_p and has the noise power spectrum Eq. (52) is called *Ohmic environment.*

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[1] A. A. Clerk *et al.*, Rev. Mod. Phys. **82,** 1155 (2010).
[2] M. H. Devoret, in Les Houches Session LXIII, *Quantum Fluctuations*, pp. 351-386 (Elsevier, Amsterdam, 1997).