Solid State Physics IV
-Part II : Macroscopic Quantum Phenomena

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We shall study LCR circuit as an example of damped harmonic oscillators. The LCR circuit can be modeled as a dissipationless LC circuit coupled to a dissipationless semi-infinite transmission line. Along this study we shall see how the more general quantum Langevin equation is emerged from the Hamiltonian formalism.

II. DAMPED HARMONIC OSCILLATORS

B. Langevin equation

1. *Circuit equation*

Let us study the situation in which an LC circuit *system* (a harmonic oscillator) coupled to a transmission line *bath* (a boson field) characterized by the impedance Z_p. The *Langevin equation* for the LC circuit is obtained by the following argument. Remembering that the right-moving voltage and the right-moving current are related as [1, 2]

$$\begin{aligned} \frac{\partial}{\partial x} V^{\rightarrow}(x,t) &= \frac{\partial}{\partial x}\dot{\varphi}^{\rightarrow}(x,t) \\ &= \frac{\partial}{\partial t}\underbrace{\frac{\partial}{\partial x}\varphi^{\rightarrow}(x,t)}_{-lI^{\rightarrow}(x,t)} = -l\left(\frac{\partial}{\partial t} I^{\rightarrow}(x,t)\right). \end{aligned} \tag{1}$$

Thus we have the current from the following expression:

$$I^{\rightarrow}(x,t) = -\frac{1}{l}\int_{-\infty}^{t} d\tau \left(\frac{\partial}{\partial x} V^{\rightarrow}(x,\tau)\right). \tag{2}$$

By plugging

$$V^{\rightarrow}(x,t) = -i\int_0^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(kx-\omega t)} - h.c.\right) \tag{3}$$

$$V^{\leftarrow}(x,t) = -i\int_0^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(-kx-\omega t)} - h.c.\right) \tag{4}$$

into the *constitutive equation* (2), which is essentially the Newton's law for the transmission line, we have

$$I^{\rightarrow}(x,t) = \frac{V^{\rightarrow}(x,t)}{Z_p} \tag{5}$$

$$I^{\leftarrow}(x,t) = -\frac{V^{\leftarrow}(x,t)}{Z_p} \tag{6}$$

Since the boundary between the transmission line bath and the LC circuit system at $x = 0$ is open we have

$$V(x=0,t) = V^{\rightarrow}(x=0,t) + V^{\leftarrow}(x=0,t) \equiv V_{out}(t) + V_{in}(t) \tag{7}$$

$$\begin{aligned} I(x=0,t) &= I^{\rightarrow}(x=0,t) + I^{\leftarrow}(x=0,t) \\ &= \frac{1}{Z_p}\left(V^{\rightarrow}(x=0,t) - V^{\leftarrow}(x=0,t)\right) \equiv \frac{1}{Z_p}\left(V_{out}(t) - V_{in}(t)\right). \end{aligned} \tag{8}$$

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This can be considered as the classical *input-output relation*. By eliminating $V_{out}(t)$ the voltage and the current relation at the boundary becomes

$$V(x=0,t) = Z_p I(x=0,t) + 2V_{in}(t). \tag{9}$$

Now let us consider the following LCR circuit equation, where the resistance stems from the coupling to the semi-infinite transmission line bath characterized by the impedance Z_p. By Kirchhoff 's law we have

$$\frac{Q(t)}{C_0} + L_0\dot{I}(x=0,t) + V(x=0,t) = 0. \tag{10}$$

With the emf voltage $V(x=0,t)$ due to the semi-infinite transmission line bath which is obtained from Eq. (9) the circuit equation becomes

$$\frac{Q(t)}{C_0} + Z_p I(x=0,t) + L_0\dot{I}(x=0,t) = -2V_{in}(t), \tag{11}$$

which leads to the following *white-noise-form* Langevin equation [3]:

$$\ddot{Q(t)} + \underbrace{\gamma}_{\frac{Z_p}{L_0}} \dot{Q(t)} + \underbrace{\omega_0}_{\frac{1}{L_0C_0}} Q(t) = -\frac{2V_{in}(t)}{L_0}, \tag{12}$$

where $V_{in}(t)$ and $Q(t)$ are the *stochastic variables*, which can be considered to have a dimension of $[\frac{1}{\sqrt{Time}}]$ and called

$$\begin{aligned} V_{in}(t) &: \text{Wiener process (white noise)} \\ Q(t) &: \text{Ornstein} - \text{Uhlenbeck process} \end{aligned}$$

respectively [3]. The above Langevin equation, Eq. (12) is a typical example of the *stochastic differential equation*, for which the more careful mathematical manipulation is required than for the ordinary differential equation [3].

Nevertheless, we shall *abuse* the Fourier transform and get

$$Q(\omega) = \frac{1}{(\omega_0^2-\omega^2) - i\omega\gamma}\left(-\frac{2V_{in}(\omega)}{L_0}\right), \tag{13}$$

which gives us the *correct* spectral density

$$S_{QQ}(\omega) = \frac{1}{(\omega_0^2-\omega^2)^2 + \omega^2\gamma^2}\left(\frac{4\bar{S}^{\leftarrow}_{VV}(\omega)}{L_0^2}\right), \tag{14}$$

where the spectral density $\bar{S}^{\leftarrow}_{VV}(\omega)$ is given by

$$\bar{S}^{\leftarrow}_{VV}(\omega) = \frac{1}{4}\bar{S}_{VV}(\omega) = Z_p\hbar\omega\left(n(\omega)+\frac{1}{2}\right) \tag{15}$$

From the *virial theorem* the capacitive energy $\langle\frac{Q^2}{2C_0}\rangle$ and inductive energy $\langle\frac{\varphi^2}{2L_0}\rangle$ share the same energy $\frac{E}{2}$. We thus have the following energy spectral density for the LCR circuit:

$$\begin{aligned} S_E(\omega) &= \frac{S_{QQ}(\omega)}{C_0} = \frac{1}{C_0}\frac{1}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\left(\frac{4Z_p\hbar\omega}{L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right) \\ &\sim \frac{1}{4\omega_0^2(\omega_0-\omega)^2+\omega_0^2\gamma^2}\left(\frac{4Z_p\hbar\omega}{C_0L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right) \\ &= \frac{1}{(\omega_0-\omega)^2+\frac{\gamma^2}{4}}\left(\frac{Z_p\hbar\omega}{\omega_0^2C_0L_0^2}\left(n(\omega)+\frac{1}{2}\right)\right) \\ &= \frac{1}{(\omega_0-\omega)^2+\frac{\gamma^2}{4}}\left(\frac{Z_p}{L_0}\hbar\omega\left(n(\omega)+\frac{1}{2}\right)\right) \\ &= \frac{\gamma}{(\omega_0-\omega)^2+\frac{\gamma^2}{4}}\left(\hbar\omega\left(n(\omega)+\frac{1}{2}\right)\right). \end{aligned} \tag{16}$$

Let us reexamine the LCR circuit from the viewpoint of Hamiltonian formalism hoping that we will gain more general tools to tackle open quantum systems.

2. *Hamiltonian formalism*

Let us reexamine the LCR circuit from the viewpoint of Hamiltonian formalism hoping that we will gain more general tools to tackle open quantum systems. Invoking the argument we have used in studying the coupled harmonic oscillators, we shall assume the total Hamiltonian to be

$$H = H_s + H_b + H_I, \tag{17}$$

where H_s, H_b, and H_I are the Hamiltonians of the LC circuit, the transmission line, and their interaction, and are respectively given by

$$\begin{aligned} H_s &= \hbar\omega_0 \hat{a}^\dagger \hat{a} && (18)\\ H_b &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hbar\omega \hat{c}^\dagger(\omega)\hat{c}(\omega) && (19)\\ H_I &= -i\hbar \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(f(\omega)\hat{a}^\dagger \hat{c}(\omega) - f^*(\omega)\hat{a}\hat{c}^\dagger(\omega) \right). && (20) \end{aligned}$$

Here the coupling strength $f(\omega)$ has the dimension of $\sqrt{\text{Angular frequency}}$ and will be identified to the familiar quantity later on.

3. *Fermi's golden rule and connecting the coupling strength f_c and the decay rate κ*

Let us begin by analyzing the coupled system by focusing on the state evolution. Suppose that the LC circuit is initially in the eigenstate $|N\rangle$ of the unperturbed LC Hamiltonian Eq. (18), i.e., $\hat{H}_s|N\rangle = \hbar\omega_0 N|N\rangle$. The time-evolution of the state $|N\rangle$ in the interaction picture can be given by

$$|N, t\rangle_I = \hat{U}_I(t)|N\rangle, \tag{21}$$

where the interaction-picture time-evolution operator $\hat{U}_I(t)$ can be gotten by the following equation:

$$i\hbar \frac{\partial}{\partial t}\hat{U}_I(t) = V_I(t)\hat{U}_I(t). \tag{22}$$

The formal solution of Eq. (22) with the initial condition of $\hat{U}_I(0) = 1$ is given by

$$\hat{U}_I(t) = 1 - \frac{i}{\hbar}\int_0^t d\tau V_I(\tau)\hat{U}_I(\tau). \tag{23}$$

The perturbative approximate solution is the famous *Dyson series* and for the current purpose we only need terms up to first order in $\hat{V}_I$:

$$\hat{U}_I(t) = 1 - \frac{i}{\hbar}\int_0^t d\tau V_I(\tau)\left(1 - \frac{i}{\hbar}\int_0^t d\tau' V_I(\tau')\left(1 - \frac{i}{\hbar}\int_0^t d\tau'' V_I(\tau'')\cdots\right)\right) \approx 1 - \frac{i}{\hbar}\int_0^t d\tau V_I(\tau). \tag{24}$$

The Schrödinger-picture interaction term H_I in Eqs. (17) and (20) and the interaction-picture interaction term V_I in Eq. (24) are related in the following way:

$$\begin{aligned} V_I(t) &= e^{i\frac{H_s+H_b}{\hbar}t} H_I e^{-i\frac{H_s+H_b}{\hbar}t}\\ &= -i\hbar\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(f(\omega)\hat{a}^\dagger e^{i\omega_0 t}\hat{c}(\omega)e^{-i\omega t} - f^*(\omega)\hat{a}e^{-i\omega_0 t}\hat{c}^\dagger(\omega)e^{i\omega t}\right). && (25) \end{aligned}$$

Let us then calculate the probability amplitude $\alpha_{N\to N+1}(t) = \langle N+1|N,t\rangle_I$ by plugging Eqs. (24) and (25) in the time evolution formula Eq. (21):

$$\begin{aligned}
\alpha_{N\to N+1}(t) &= \langle N+1|N,t\rangle_I \\
&= \langle N+1|\hat{U}_I(t)|N\rangle \\
&= \underbrace{\langle N+1|N\rangle}_{0} - \frac{i}{\hbar}\int_0^t d\tau \langle N+1|\hat{V}_I(\tau)|N\rangle \\
&= -\int_0^t d\tau \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(f(\omega)\underbrace{\langle N+1|\hat{a}^\dagger|N\rangle}_{\sqrt{N+1}} e^{i\omega_0\tau}\hat{c}(\omega)e^{-i\omega\tau} - f^*(\omega)\underbrace{\langle N+1|\hat{a}|N\rangle}_{0} e^{-i\omega_0\tau}\hat{c}^\dagger(\omega)e^{i\omega\tau}\right) \\
&= -\sqrt{N+1}\int_0^t d\tau e^{i\omega_0\tau}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}f(\omega)\hat{c}(\omega)e^{-i\omega\tau}. \qquad (26)
\end{aligned}$$

When our concerned frequency band is narrow and centered on some peculiar frequency $\frac{\Omega}{2\pi}$ we could make the *Markov approximation*, in which the coupling strength is assumed to be frequency-independent, i.e., $f(\omega) \sim f(\Omega) = f_c$:

$$\alpha_{N\to N+1}(t) = -\sqrt{N+1}\ f_c \int_0^t d\tau e^{i\omega_0\tau}\underbrace{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega)e^{-i\omega\tau}}_{\hat{c}(t)e^{-i\Omega\tau}}. \qquad (27)$$

Here the time-domain operator $\hat{c}(t)$ and its conjugate $\hat{c}^\dagger(t)$ are introduced, which are within the Markov approximation can be characterized by the following relation:

$$\begin{aligned}
[\hat{c}(t), \hat{c}^\dagger(t')] &= \delta(t-t') \qquad (28)\\
\langle \hat{c}^\dagger(t)\hat{c}(t')\rangle &= n(\Omega)\delta(t-t'), \qquad (29)\\
\langle \hat{c}(t)\hat{c}^\dagger(t')\rangle &= (n(\Omega)+1)\,\delta(t-t'), \qquad (30)
\end{aligned}$$

which can be considered as the characteristics of the quantum analogue of white noise. Note that the average photon number $n(\Omega)$ is a *dimensionless* quantity, which can be obtained by

$$n(\Omega) = \int_{\Omega-\frac{\Delta}{2}}^{\Omega+\frac{\Delta}{2}} d\omega \langle \hat{c}^\dagger(\omega)\hat{c}(\omega)\rangle, \qquad (31)$$

where Δ is the bandwidth of the concerned bath mode.

The probability that state of the LC oscillator changes from unperturbed eigenstate $|N\rangle$ to unperturbed eigenstate $|N+1\rangle$ is

$$\begin{aligned}
P_{N\to N+1}(t) &= |\alpha_{N\to N+1}(t)|^2 \\
&= |f_c|^2(N+1)\int_0^t d\tau_1 e^{-i(\omega_0-\Omega)\tau_1}\hat{c}^\dagger(\tau_1)\int_0^t d\tau_2 e^{i(\omega_0-\Omega)\tau_2}\hat{c}(\tau_2) \\
&= |f_c|^2(N+1)\int_0^t d\tau_2\int_{-\tau_2}^{t-\tau_2} dT\left(\hat{c}^\dagger(T+\tau_2)\hat{c}(\tau_2)\right), \qquad (32)
\end{aligned}$$

where $T = \tau_1 - \tau_2$ and $\omega_0 = \Omega$ is assumed. The auto-correlation term $\hat{c}^\dagger(T+\tau_2)\hat{c}(\tau_2)$ is assumed to be stationary (i.e., independent on τ_2) thus by taking the average over τ_2 we have $\langle \hat{c}^\dagger(T+\tau_2)\hat{c}(\tau_2)\rangle = \bar{n}(\omega_0)\delta(T)$, which has negligible value beyond the characteristic correlation time $T > \tau_c$. Then we can extend the both bounds of second integral to infinity and we have

$$\begin{aligned}
P_{N\to N+1}(t) &= |f_c|^2(N+1)\int_0^t d\tau_2\int_{-\infty}^{\infty} dT\underbrace{\langle \hat{c}^\dagger(T)\hat{c}(0)\rangle}_{n(\omega_0)\delta(T)} \\
&= |f_c|^2(N+1)\ t\ n(\omega_0), \qquad (33)
\end{aligned}$$

From Eq. (33) we see that the probability $P_{N\to N+1}(t)$ grows linearly with time t. The transition rate for $N \to N+1$ can then be given by the time derivative of $P_{N\to N+1}(t)$, i.e.,

$$\Gamma_{N\to N+1} = \frac{dP_{N\to N+1}(t)}{dt} = (N+1)\underbrace{|f_c|^2 n(\omega_0)}_{\Gamma_\uparrow} = (N+1)\Gamma_\uparrow. \qquad (34)$$

This is the upward transition rate of a harmonic oscillator essentially leading to the same conclusion as *Fermi's golden rule*. Similarly, we have the downward transition rate for $N \to N-1$

$$\Gamma_{N\to N-1} = N \underbrace{|f_c|^2 \left(n(\omega_0)+1\right)}_{\Gamma_\downarrow} = N\Gamma_\downarrow. \tag{35}$$

Let us now see the consequence of the system-bath coupling. The photon occupation number probability $P_N(t)$ for the LC circuit system changes with time due to the coupling to the bath. Its rate of change $\frac{P(N,t)}{dt}$ can be determined by the four terms as follow

$$\frac{P(N,t)}{dt} = \Gamma_\uparrow NP(N-1) - \Gamma_\uparrow(N+1)P(N) - \Gamma_\downarrow NP(N) + \Gamma_\downarrow(N+1)P(N+1), \tag{36}$$

where $\Gamma_\uparrow$ and $\Gamma_\downarrow$ are given by Fermi's golden rule, that is, by Eq. (34) and (35), respectively. Consequently we have the following rate equation:

$$\begin{aligned}
\frac{d}{dt}\langle N\rangle &= \sum_N N\frac{P(N,t)}{dt} \\
&= \Gamma_\uparrow \sum_N (N+1)P(N) - \Gamma_\downarrow \sum_N NP(N), \\
&= \Gamma_\uparrow\langle N+1\rangle - \Gamma_\downarrow\langle N\rangle \\
&= -\left(\Gamma_\downarrow - \Gamma_\uparrow\right)\langle N\rangle + \Gamma_\uparrow \\
&= -|f_c|^2\langle N\rangle + |f_c|^2 n(\omega_0) \\
&= -\kappa\langle N\rangle + \kappa n(\omega_0),
\end{aligned} \tag{37}$$

where we used Eqs. (34) and (35) for the last line. The steady state condition $\frac{d}{dt}\langle N\rangle = 0$ gives us the expected final occupation number of the LC photons:

$$\bar{N} = n(\omega_0), \tag{38}$$

that is, the averaged photon number of the bath. The solution of the rate equation Eq. (37) can then be given by

$$\langle N(t)\rangle = \langle N(0)\rangle e^{-\kappa t} + n(\omega_0)\left(1 - e^{-\kappa t}\right). \tag{39}$$

We thus find that the coupling constant f_c and the decay rate of the LC photon κ are related as $f_c = \sqrt{\kappa}$.

4. Quantum Langevin equation and connecting the two decay rate κ and $\gamma = \frac{Z_p}{L_0}$

Now that we identify the coupling strength $f(\omega)$ in Eq. (20) as $f(\omega) \sim f_c = \sqrt{\kappa}$ within the Markov approximation, we shall investigate the couped Heisenberg equations derived from the following interaction Hamiltonian:

$$H_I = -i\hbar\sqrt{\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega) - \hat{a}\hat{c}^\dagger(\omega)\right). \tag{40}$$

Note that in the interaction picture the Hamiltonian Eq. (40) can be made further simpler,

$$\tilde{H}_i - i\hbar\sqrt{\kappa}\left(\hat{a}^\dagger\hat{c}(t)e^{-i\Omega t} - \hat{a}\hat{c}^\dagger(t)e^{i\Omega t}\right), \tag{41}$$

with the time-domain operators $\hat{c}(t)$ and $\hat{c}^\dagger(t)$. The Heisenberg equation of motion for the bath is

$$\dot{\hat{c}}(\omega,t) = \frac{i}{\hbar}\left[H, \hat{c}(\omega,t)\right] = -i\omega\hat{c}(\omega,t) + \sqrt{\kappa}\hat{a}(t). \tag{42}$$

We can find the formal solution of Eq. (42) as

$$\hat{c}(\omega,t) = e^{-i\omega(t-t_0)}\hat{c}(\omega,t_0) + \sqrt{\kappa}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau). \tag{43}$$

The Heisenberg equation of motion for the system, on the other hand, is given by

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}\left[H, \hat{a}(t)\right] = -i\omega_0 \hat{a}(t) - \sqrt{\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega, t). \tag{44}$$

By plugging the solution for $\hat{c}(\omega, t)$ in Eq. (43) into Eq. (44) we have

$$\begin{aligned}
\dot{\hat{a}}(t) &= -i\omega_0 \hat{a}(t) - \sqrt{\kappa}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0) + \sqrt{\kappa}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau)\right) \\
&= -i\omega_0 \hat{a}(t) - \sqrt{\kappa}\underbrace{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0)}_{\hat{c}(t)e^{-i\Omega t}} - \kappa\underbrace{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau)}_{\int_{t_0}^{t} d\tau\hat{a}(\tau)\delta(t-\tau)=\frac{1}{2}\hat{a}(t)} \\
&= \underbrace{-i\omega_0 \hat{a}(t)}_{\text{Free evolution}} \underbrace{-\sqrt{\kappa}\hat{c}(t)e^{-i\Omega t}}_{\text{Fluctuation}} \underbrace{-\frac{\kappa}{2}\hat{a}(t)}_{\text{Dissipation}},
\end{aligned} \tag{45}$$

which shall be called the *quantum Langevin equation.*

Let $\hat{\alpha}(t)$ be defined by $\hat{a}(t) = \hat{\alpha}(t)e^{-i\Omega t}$. Plugging this $\hat{\alpha}(t)$ in Eq. (45) we have

$$\dot{\hat{\alpha}}(t) = -i\left(\omega_0 - \Omega\right)\hat{\alpha}(t) - \sqrt{\kappa}\hat{c}(t) - \frac{\kappa}{2}\hat{\alpha}(t). \tag{46}$$

The steady state solution for $\hat{\alpha}(t)$ is

$$\hat{\alpha}(t) = \frac{-\sqrt{\kappa}}{i\left(\omega_0 - \Omega\right) + \frac{\kappa}{2}}\hat{c}(t), \tag{47}$$

thus we have the average photon number in the LC circuit:

$$\begin{aligned}
\langle\hat{\alpha}^{\dagger}(t)\hat{\alpha}(t')\rangle &= \frac{\kappa}{\left(\omega_0 - \Omega\right)^2 + \frac{\kappa^2}{4}}\langle\hat{c}^{\dagger}(t)\hat{c}(t')\rangle \\
&= \frac{\kappa}{\left(\omega_0 - \Omega\right)^2 + \frac{\kappa^2}{4}}n(\Omega)\delta(t - t'),
\end{aligned} \tag{48}$$

where Eq. (29) is used for the last equation. Consequently, the spectral density at the angular frequency Ω for the LC photon can be obtained by

$$\begin{aligned}
S_n(\Omega) &= \int_{-\infty}^{\infty} dT\langle\hat{\alpha}^{\dagger}(T)\hat{\alpha}(0)\rangle e^{i\Omega T} \\
&= \int_{-\infty}^{\infty} dT\frac{\kappa}{\left(\omega_0 - \Omega\right)^2 + \frac{\kappa^2}{4}}n(\Omega)\delta(T)e^{i\Omega T} \\
&= \frac{\kappa}{\left(\omega_0 - \Omega\right)^2 + \frac{\kappa^2}{4}}n(\Omega).
\end{aligned} \tag{49}$$

Lo and behold, by replacing κ with $\gamma = \frac{Z_p}{L_0}$ and multiplying the unit energy $\hbar\Omega$, we can reproduce the energy spectral density Eq. (16), which was obtained by the more explicit argument with the circuit equation.

Now the story comes full circle. The coupling of the LC circuit system to the semi-infinite (dissipative) transmission line bath can be modeled by the interaction

$$H_I = -i\hbar\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sqrt{\frac{Z_p}{L_0}}\left(\hat{a}^{\dagger}\hat{c}(\omega) - (\omega)\hat{a}\hat{c}^{\dagger}(\omega)\right), \tag{50}$$

where Z_p is the impedance of the bath and L_0 is the inductance of the system. We thus succeed in treating the open dissipative system quantum mechanically!

The quantum Langevin equation (45) is one of the most useful equation in quantum optics and in treating macroscopic quantum phenomena, which is applicable to many other open quantum systems where a (0+1)-dimensional system coupled to a continuum (d+1)-dimensional bath, where d is the spatial dimension of the bath. The extension to the coupling between a two level system and (3+1)-dimensional electromagnetic environments leads us to the realm of the *cavity QED*, which will be our future subject.

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