Solid State Physics IV
-Part II : Macroscopic Quantum Phenomena

Koji Usami*

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We deduce the useful *input-output theory* from the Hamiltonian formalism. The input-output theory combined with the quantum Langevin equation is the basic tool to formulate the dissipative quantum systems, which are coupled to the Markovian bath. We shall then move on to the discussion of how the anharmonicity plays the important role in quantum mechanics, which leads us to the new basic player in the quantum world, the *two level system*, and to the realm of cavity QED, where the two level system is strongly interacting with a single boson mode as well as being weakly interacting with a bunch of other boson modes.

II. DAMPED HARMONIC OSCILLATORS

C. The input-output theory

Let us examine the quantum Langevin equation a bit further. Let the system-bath Hamiltonian be

$$H = H_s + H_b + H_I, \tag{1}$$

with

$$\begin{aligned} H_s &= H_s(\hat{a}, \hat{a}^\dagger, \hat{b}(\omega), \hat{b}^\dagger(\omega), \cdots) && (2) \\ H_b &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hbar\omega \hat{c}^\dagger(\omega)\hat{c}(\omega) && (3) \\ H_I &= -i\hbar\sqrt{\kappa} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\hat{a}^\dagger \hat{c}(\omega) - \hat{a}\hat{c}^\dagger(\omega) \right), && (4) \end{aligned}$$

where the system Hamiltonian now contains the other field operators $\hat{b}(\omega), \hat{b}^\dagger(\omega), \cdots$ suggesting the existence of other decay channels.

We are interested in the effect of the bath mode specified by the operators $\hat{c}(\omega)$ and $\hat{c}^\dagger(\omega)$ on the system which interacts not only the concerned bath mode but also the other bath modes. The equation of motion for the bath mode is the same as before:

$$\dot{\hat{c}}(\omega, t) = -i\omega\hat{c}(\omega, t) + \sqrt{\kappa}\hat{a}(t), \tag{5}$$

while that for the system becomes

$$\dot{\hat{a}}(t) = \frac{i}{\hbar} [H_s, \hat{a}(t)] - \sqrt{\kappa} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} c(\omega, t). \tag{6}$$

We can find two formal solutions for Eq. (5); one of which is the one we have already encountered and we shall call it the *input mode*,

$$\hat{c}_{in}(\omega, t) = e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0) + \sqrt{\kappa} \int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)\hat{a}(\tau)}, \tag{7}$$

which is defined by referring to the *past time* t_0, the other is the *output mode*,

$$\hat{c}_{out}(\omega, t) = e^{-i\omega(t-t_1)}\hat{c}(\omega, t_1) - \sqrt{\kappa} \int_{t}^{t_1} d\tau e^{-i\omega(t-\tau)\hat{a}(\tau)}, \tag{8}$$

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*Electronic address: usami@qc.rcast.u-tokyo.ac.jp

which is defined by referring to the *future time* t_1. Plugging those solutions in Eq. (6) we have two equations of motions for the system:

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[H_s, \hat{a}(t)] - \frac{\kappa}{2}\hat{a}(t) - \sqrt{\kappa}\hat{c}_{in}(t)e^{-i\Omega t} \tag{9}$$

$$\dot{\hat{a}}(t) = \frac{i}{\hbar}[H_s, \hat{a}(t)] + \frac{\kappa}{2}\hat{a}(t) - \sqrt{\kappa}\hat{c}_{out}(t)e^{-i\Omega t}, \tag{10}$$

where the time-domain operators $\hat{c}_{in}(t)$ and $\hat{c}_{out}(t)$ within the Markov approximation are defined by

$$\hat{c}_{in}(t)e^{-i\Omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0) \tag{11}$$

$$\hat{c}_{out}(t)e^{-i\Omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t_1)}\hat{c}(\omega, t_1), \tag{12}$$

respectively. By subtracting Eq. (10) from Eq. (9) we have

$$0 = -\kappa\hat{a}(t) - \sqrt{\kappa}\hat{c}_{in}(t)e^{-i\Omega t} + \sqrt{\kappa}\hat{c}_{out}(t)e^{-i\Omega t} \tag{13}$$

which leads to the very useful *input-output relation* [1]:

$$\hat{c}_{out}(t) = \hat{c}_{in}(t) + \sqrt{\kappa}\hat{\alpha}(t) \tag{14}$$

with $\hat{a}(t) = \hat{\alpha}(t)e^{-i\Omega t}$. The input-output relation (14) can be compared with the more explicit *classical input-output relation* for the LC circuit with 1D transmission line:

$$V_{out}(t) = V_{in}(t) + Z_p I(t), \tag{15}$$

which can be rewritten as

$$\frac{V_{out}(t)}{\sqrt{Z_p}} = \frac{V_{in}(t)}{\sqrt{Z_p}} + \sqrt{Z_p} I(t), \tag{16}$$

or more suggestive form with the flux variable $\varphi(t) = L_0 I(t)$:

$$\underbrace{\frac{V_{out}(t)}{\sqrt{Z_p}}}_{\sim\sqrt{\hbar\Omega}\hat{c}_{out}(t)} = \underbrace{\frac{V_{in}(t)}{\sqrt{Z_p}}}_{\sim\sqrt{\hbar\Omega}\hat{c}_{in}(t)} + \underbrace{\sqrt{\frac{Z_p}{L_0}}\frac{\varphi(t)}{\sqrt{L_0}}}_{\sim\sqrt{\kappa}\sqrt{\hbar\Omega}\hat{a}(t)}. \tag{17}$$

The system which is coupled to the Markovian bath with the decay rate κ can be fully analyzed with the equation of motion (9) and the input-output relation (14).

III. TWO LEVEL SYSTEM AND CAVITY QED

A. Electromagnetic field in a cavity [3, 4]

Before studying the two level system, the yet another major player in quantum optics, let us review how the electromagnetic field in a cavity can be treated quantum mechanically, which acts as the environment and induces the dissipation for the two level system. This section can be considered as the 3D vector extension of the 1D scalar transmission line modes specified by the flux $\varphi(k)$ and charge $q(k)$ we have studied before.

The free field Hamiltonian in the *Coulomb gauge* can then be given by [4]

$$H_R = \frac{\epsilon_0}{2}\int_{\text{cavity}} dV \left(\underbrace{\left(\frac{\boldsymbol{\Pi}(\boldsymbol{r})}{\epsilon_0}\right)^2}_{\boldsymbol{E}_\perp(\boldsymbol{r})^2} + c^2 \underbrace{\left(\nabla \times \boldsymbol{A}(\boldsymbol{r})\right)^2}_{\boldsymbol{B}_\perp(\boldsymbol{r})^2} \right). \tag{18}$$

In reciprocal space the Hamiltonian becomes

$$\begin{aligned} H_R &= \sum_{\boldsymbol{k}} \sum_{\lambda=1,2} \frac{\epsilon_0}{2} \left(\left(\frac{\boldsymbol{\Pi}_{\lambda,\boldsymbol{k}}}{\epsilon_0} \cdot \frac{\boldsymbol{\Pi}_{\lambda,-\boldsymbol{k}}}{\epsilon_0} \right) + c^2 k^2 \left(\boldsymbol{A}_{\lambda,\boldsymbol{k}} \cdot \boldsymbol{A}_{\lambda,-\boldsymbol{k}} \right) \right) \\ &= \sum_k \mathcal{H}_k \end{aligned} \tag{19}$$

where $\mathcal{H}_k$ is the Hamiltonian for a *single mode* with the index $k = \{\boldsymbol{k}, \lambda\}$:

$$\mathcal{H}_{\lambda,\boldsymbol{k}} = \mathcal{H}_k = \underbrace{\frac{1}{2\epsilon_0} \Pi_k \cdot \Pi_{-k}}_{\text{kinetic part}} + \underbrace{\frac{1}{2} \epsilon_0 \omega_k^2 A_k \cdot A_{-k}}_{\text{potential part}}, \tag{20}$$

where $\omega_k = ck$ and the Fourier transforms (with the first continuum limit) are given by

$$\boldsymbol{\Pi}_{\lambda,\boldsymbol{k}} = \boldsymbol{e}_{\lambda,\boldsymbol{k}} \Pi_k = \frac{\boldsymbol{e}_{\lambda,\boldsymbol{k}}}{\sqrt{V}} \int_{cavity} dV \left(\boldsymbol{e}_{\lambda,\boldsymbol{k}} \cdot \boldsymbol{\Pi}(\boldsymbol{r}) \right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{21}$$

$$\boldsymbol{A}_{\lambda,\boldsymbol{k}} = \boldsymbol{e}_{\lambda,\boldsymbol{k}} A_k = \frac{\boldsymbol{e}_{\lambda,\boldsymbol{k}}}{\sqrt{V}} \int_{cavity} dV \left(\boldsymbol{e}_{\lambda,\boldsymbol{k}} \cdot \boldsymbol{A}(\boldsymbol{r}) \right) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{22}$$

The form of Eq. (19) is the same as that for harmonic oscillators. In this sense we can consider the single-mode free electromagnetic field in the Coulomb gauge (in reciprocal space) as a harmonic oscillator with the position variable $A(\omega)$, the momentum variable $\Pi(\omega)$, and the mass ϵ_0 . The commutation relation is still with Kronecker delta:

$$[A_k, \Pi_{k'}] = i\hbar \delta_{k,k'} \tag{23}$$

With the annihilation and creation operators

$$\hat{a}_k = \sqrt{\frac{\epsilon_0 \omega_k}{2\hbar}} \left(\hat{A}_{-k} + \frac{i}{\epsilon_0 \omega_k} \hat{\Pi}_k \right) \tag{24}$$

$$\hat{a}_k^\dagger = \sqrt{\frac{\epsilon_0 \omega_k}{2\hbar}} \left(\hat{A}_k - \frac{i}{\epsilon_0 \omega_k} \hat{\Pi}_{-k} \right), \tag{25}$$

the Hamiltonian Eq. (19) can be rewritten in a diagonal form:

$$H_R = \sum_k \mathcal{H}_k = \sum_k \hbar \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right). \tag{26}$$

From Eqs. (24) and (25) we have

$$\Pi_k = -i \sqrt{\frac{\epsilon_0 \hbar \omega_k}{2}} \left(\hat{a}_k - \hat{a}^\dagger_{-k} \right) \tag{27}$$

$$A_k = \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k}} \left(\hat{a}_{-k} + \hat{a}_k^\dagger \right). \tag{28}$$

The time evolutions of $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ are dictated by the free Hamiltonian (26), which leads to

$$\hat{a}_k(t) = \hat{a}_k(0) e^{-i\omega_k t} \tag{29}$$

$$\hat{a}_k^\dagger(t) = \hat{a}_k^\dagger(0) e^{i\omega_k t} \tag{30}$$

and thus

$$\Pi_k(t) = -i \sqrt{\frac{\epsilon_0 \hbar \omega_k}{2}} \left(\hat{a}_k(0) e^{-i\omega_k t} - \hat{a}^\dagger_{-k}(0) e^{i\omega_k t} \right). \tag{31}$$

The real space operator $\mathbf{\Pi}(\boldsymbol{r},t)$ can then be given by the Fourier transformation of $\Pi_k(t)$:

$$\begin{aligned}\mathbf{\Pi}(\boldsymbol{r},t) &= \frac{1}{\sqrt{V}}\sum_k \boldsymbol{e}_{\lambda,\boldsymbol{k}}\Pi_k(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= -i\sum_k\sqrt{\frac{\epsilon_0\hbar\omega_k}{2V}}\boldsymbol{e}_{\lambda,\boldsymbol{k}}\left(\hat{a}_k(0)e^{-i\omega_k t}-\hat{a}^\dagger_{-k}(0)e^{i\omega_k t}\right)e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= -i\sum_k\sqrt{\frac{\epsilon_0\hbar\omega_k}{2V}}\boldsymbol{e}_{\lambda,\boldsymbol{k}}\left(\hat{a}_k(0)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-\hat{a}^\dagger_k(0)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\right). \end{aligned} \tag{32}$$

Consequently we have the transverse electric field operator

$$\boldsymbol{E}_\perp(\boldsymbol{r},t) = -\frac{\mathbf{\Pi}(\boldsymbol{r},t)}{\epsilon_0} = -\dot{\boldsymbol{A}}(\boldsymbol{r},t) = i\sum_k\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\boldsymbol{e}_{\lambda,\boldsymbol{k}}\left(\hat{a}_k(0)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}-h.c.\right). \tag{33}$$

Similarly the real space vector potential can be obtained as

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_k\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}\boldsymbol{e}_{\lambda,\boldsymbol{k}}\left(\hat{a}_k(0)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}+h.c.\right), \tag{34}$$

and thus the magnetic field operator becomes

$$\boldsymbol{B}(\boldsymbol{r},t) = \nabla\times\boldsymbol{A}(\boldsymbol{r},t) = \sum_k\sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}}\left(\boldsymbol{k}\times\boldsymbol{e}_{\lambda,\boldsymbol{k}}\right)\left(\hat{a}_k(0)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}+h.c.\right). \tag{35}$$

B. Two level systems

We have been so far exclusively studying harmonic oscillators, which are always in the *correspondence limit* displaying only mundane quantum effects [2]. The average value of the generalized position and momentum follows the classical equations of motion and quantum mechanical features manifest themselves in higher moments like the variance of those basic quantities. To see more direct and interesting macroscopic quantum effects it requires at least one non-linear component which should be workable in the quantum regime. A very basic non-linear component in the microscopic world is an atom, and that in the macroscopic world is an *artificial atom with Josephson tunnel junctions*. The non-linear component brings in the *anharmonicity* in the system and brings about the *saturation phenomena*, for example.

1. *Anharmonic potential and energy level spacing [5]*

Let us begin by examining the energy level spacing for the system with the anharmonic potential. Imagine the particle with mass m moving under the 1-dimensional potential $U(x) = A|x|^k$. Since the energy is given by

$$E = \frac{m}{2}\dot{x}^2 + U(x), \tag{36}$$

we have

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}(E-U(x))}. \tag{37}$$

Integration of Eq. (37) leads to

$$t = \sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E-U(x)}} + const. \tag{38}$$

From Eq. (38) the oscillation period for the particle trapped in the potential $U(x) = A|x|^k$ can be found to be

$$
\begin{aligned}
T &= 4\sqrt{\frac{m}{2}}\int_0^{\left(\frac{E}{A}\right)^{\frac{1}{k}}} \frac{dx}{\sqrt{E - Ax^k}} \\
&= 2\sqrt{2m}A^{-\frac{1}{k}}E^{\frac{1}{k}-\frac{1}{2}}\int_0^1 \frac{dy}{\sqrt{1-y^k}} \\
&\propto E^{\frac{1}{k}-\frac{1}{2}},
\end{aligned}
\tag{39}
$$

where the upper limit of the integral $\left(\frac{E}{A}\right)^{\frac{1}{k}}$ in the first line comes from the turning-point condition $E = Ax^k$. We thus recognize the peculiarity of the harmonic potential ($k = 2$), that is, $T = const.$, or rather

$$
\Delta E = \hbar\omega = \hbar\frac{2\pi}{T} = const. \tag{40}
$$

On the other hand, the oscillation periods for anharmonic potentials depend on their energies. For instance, the oscillation period for a potential $U(x) = -A|x|^{-1}$ responsible associated with the *inverse square law* $F(x) = A|x|^{-2}$ is $T \propto (-E)^{-\frac{3}{2}}$ (note that the minus sign before E coming from the fact that the E is *negative* for the case of $k < 0$), that for a triangular potential $U(x) = A|x|$ is $T \propto E^{\frac{1}{2}}$, and that for the infinitely deep wall potential $U(x) = \left(\frac{x}{l}\right)^{\infty}$ is $T \propto E^{-\frac{1}{2}}$. These findings lead to

$$
\begin{cases}
\Delta E_{inv} = \hbar\omega = \hbar\frac{2\pi}{T} \propto (-E)^{\frac{3}{2}} & \text{for } U(x) = -A|x|^{-1} \\
\Delta E_{tri} = \hbar\omega = \hbar\frac{2\pi}{T} \propto E^{-\frac{1}{2}} & \text{for } U(x) = A|x| \\
\Delta E_{wall} = \hbar\omega = \hbar\frac{2\pi}{T} \propto E^{\frac{1}{2}} & \text{for } U(x) = \left(\frac{x}{l}\right)^{\infty}.
\end{cases}
\tag{41}
$$

2. *Bohr's atom [6, 7]*

Now let us see how does Bohr's atom exemplify the anharmonic quantum paradigm. Suppose that an electron is orbiting the atomic nucleus in a circle. The radius r is determined by the balance between the *centrifugal force* $m_e\boldsymbol{\Omega}\times(\boldsymbol{r}\times\boldsymbol{\Omega}) = \frac{m_e v^2}{r}\boldsymbol{e_r}$ and the *inverse-square-law Coulomb force* $\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}\boldsymbol{e_r}$, that is,

$$
\frac{m_e v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}. \tag{42}
$$

The total energy is given by

$$
E = \underbrace{\frac{1}{2}m_e v^2}_{\text{Kinetic energy:T}} + \underbrace{-\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}}_{\text{Potential energy:U}}. \tag{43}
$$

Since from the *virial theorem* the average kinetic energy $\bar{T}$ can be obtained by

$$
2\bar{T} = k\bar{U} \tag{44}
$$

for the potential $U(r) = -Ar^k$, we have

$$
\begin{aligned}
\bar{T} &\equiv \frac{1}{2}m_e\bar{v^2} \\
&= (-1)\frac{\bar{U}}{2} \equiv \frac{1}{4\pi\epsilon_0}\frac{e^2}{2\bar{r}},
\end{aligned}
\tag{45}
$$

whose validity can also be checked by Eq. (42). The average total energy can then be written as

$$
\bar{E} = \bar{T} + \bar{U} = \frac{\bar{U}}{2} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r}. \tag{46}
$$

The quantization of the angular momentum further impose

$$
m_e v r = n\hbar. \tag{47}
$$

Using Eqs. (42) and (47) the *quantized radius* can be obtained as

$$r_n = \frac{\hbar^2}{\frac{e^2}{4\pi\epsilon_0}m_e}n^2 = a_0 n^2, \tag{48}$$

where

$$a_0 = \frac{\hbar^2}{\frac{e^2}{4\pi\epsilon_0}m_e} \sim 0.53 \times 10^{-10}\ \mathrm{m} \tag{49}$$

is the *Bohr radius*, a typical length scale in atomic physics, which is made up of fundamental constants. Plugging Eq. (48) into Eq. (43) leads to the famous Bohr formula:

$$E = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a_0}\frac{1}{n^2} = -\frac{Ry}{n^2}, \tag{50}$$

where

$$Ry \equiv 2\pi\hbar c R_\infty = \frac{1}{4\pi\epsilon_0}\frac{e^2}{2a_0} = \frac{e^4 m_e}{8(2\pi\hbar)^2\epsilon_0^2} \sim 13.6\ \mathrm{eV} \tag{51}$$

is the *Rydberg characteristic energy*, a typical energy scale in atomic physics, which is again made up of fundamental constants.

Let us now see the anharmonic nature of Bohr's atom. For $n \gg 1$ the energy spacing ΔE_n between E_{n+1} and E_n is given by

$$\begin{aligned}
\Delta E_n = E_{n+1} - E_n &= -Ry\left(\left(\frac{1}{n+1}\right)^2 - \left(\frac{1}{n}\right)^2\right) \\
&\sim \frac{2Ry}{n^2(n+2)} \\
&\sim \frac{2Ry}{n^3}\left(= \frac{\partial E}{\partial n}\right) \\
&= \frac{2Ry}{\left(-\frac{Ry}{E_n}\right)^{\frac{3}{2}}} \propto (-E_n)^{\frac{3}{2}}\,,
\end{aligned} \tag{52}$$

which agrees with the general classical result Eq. (41) for the potential associated with the inverse-square-law force.

We shall now introduce one more useful constant, the *fine-structure constant*:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{e^2}{2\epsilon_0 hc} = \frac{1}{2}\left(\frac{1}{\frac{h}{e^2}}\right)\left(\frac{1}{c\epsilon_0}\right) = \frac{1}{2}\frac{Z_{vac}}{R_k} \sim \frac{1}{137}, \tag{53}$$

where

$$R_k = \frac{h}{e^2} \sim 25.8\ \mathrm{k}\Omega \tag{54}$$

$$Z_{vac} = \frac{1}{c\epsilon_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \sim 377\ \mathrm{k}\Omega \tag{55}$$

are the *impedance quantum* and the *impedance of vacuum*, respectively. With the fine-structure constant α the Bohr radius a_0 and the Rydberg characteristic energy Ry can be rewritten as more meaningful forms whose dimensions manifest themselves with fundamental constants:

$$a_0 = \frac{1}{\alpha}\frac{\hbar}{m_e c} \tag{56}$$

$$Ry = \frac{\alpha^2}{2}m_e c^2. \tag{57}$$

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