Solid State Physics IV
-Part II : Macroscopic Quantum Phenomena

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We shall look at how two level systems can be treated quantum mechanically, learn how the electric dipole interaction emerged as the coupling term between a two level system and electromagnetic field, and deduce Einstein's A coefficient, a decay rate of the two level system due to the coupling to the *vacuum fluctuation* of electromagnetic field surrounding it. We then learn how a cavity changes the *density of state* of the electromagnetic environment and modifies the decay rate of two level systems.

III. TWO LEVEL SYSTEM AND CAVITY QED

B. Two level systems

3. Quantum description of two level systems

The system with anharmonic potential can be treated as the *two level system* by exploiting the fact that each energy spacing is unique thus by choosing two *distinct* energy levels $|0\rangle$ and $|1\rangle$, which are the energy eigenstates with energies $\hbar\omega_0$ and $\hbar\omega_1 > \hbar\omega_0$, respectively. The useful operators for the two level systems are the Pauli operators,

$$\hat{\sigma}_0 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{1}$$

$$\hat{\sigma}_x = \frac{1}{2}\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{2}$$

$$\hat{\sigma}_y = \frac{i}{2}\left(-|1\rangle\langle 0| + |0\rangle\langle 1|\right) = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \tag{3}$$

$$\hat{\sigma}_z = \frac{1}{2}\left(-|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{4}$$

which satisfy the angular momentum commutation relation

$$[\hat{\sigma}_i, \hat{\sigma}_j] = i\epsilon_{ijk}\hat{\sigma}_k \tag{5}$$

The Pauli operators in the form of Eqs. (1), (2), (3), and (4) thus play the role of generators of rotation. The ladder operators are then defined by

$$\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y. \tag{6}$$

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The *irreducible rank-1 tensor operators* [1] can constructed with the Pauli operators,

$$\begin{aligned} T_1^1 &= -\frac{1}{\sqrt{2}}\left(\hat{\sigma}_x + i\hat{\sigma}_y\right) \\ &= -\frac{1}{\sqrt{2}}|1\rangle\langle 0| = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{7}$$

$$\begin{aligned} T_0^1 &= \hat{\sigma}_z \\ &= -\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \end{aligned} \tag{8}$$

$$\begin{aligned} T_{-1}^1 &= \frac{1}{\sqrt{2}}\left(\hat{\sigma}_x - i\hat{\sigma}_y\right) \\ &= \frac{1}{\sqrt{2}}|0\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

to satisfy the definition of the irreducible tensors

$$\begin{aligned} \left[\hat{\sigma}_z, T_q^k\right] &= q\, T_q^k \\ \left[\hat{\sigma}_\pm, T_q^k\right] &= \sqrt{k(k+1) - q(q\pm 1)}\; T_{q\pm 1}^k, \end{aligned} \tag{10}$$

which can be viewed as an operator extension of the more familiar definition of angular momentum eigenstates

$$\begin{aligned} \hat{\sigma}_z|k,q\rangle &= q\; |k,q\rangle \\ \hat{\sigma}_\pm|k,q\rangle &= \sqrt{k(k+1) - q(q\pm 1)}\; |k,q\pm 1\rangle. \end{aligned} \tag{11}$$

The Hamiltonian of an electron in Coulomb potential $U(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$ can be written in terms of Pauli operators:

$$\begin{aligned} H_a &= \frac{\boldsymbol{p}^2}{2m_e} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\ &\simeq \hbar\omega_0|0\rangle\langle 0| + \hbar\omega_1|1\rangle\langle 1| \\ &= \hbar\underbrace{(\omega_1 + \omega_0)}_{\Omega}\hat{\sigma}_0 + \hbar\underbrace{(\omega_1 - \omega_0)}_{\omega_A}\hat{\sigma}_z \\ &= \hbar\Omega\hat{\sigma}_0 + \hbar\omega_A\hat{\sigma}_z. \end{aligned} \tag{12}$$

The electric dipole moment of Bohr's atom can be written in terms of Pauli operators

$$\boldsymbol{d} = e\boldsymbol{r} = er_n\boldsymbol{e}_r = ea_0n^2\left(\hat{\sigma}_x\boldsymbol{e}_x + \hat{\sigma}_y\boldsymbol{e}_y + \hat{\sigma}_z\boldsymbol{e}_z\right) \tag{13}$$

with the Cartesian basis $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$. It should be noted that the electric dipole operator $\boldsymbol{d}$ is the *polar vector* operator having odd parity. By appreciating the characteristic symmetry property of the polar vector, the electric dipole moment $\boldsymbol{d}$ can be given in terms of the irreducible rank-1 tensor operators

$$\boldsymbol{d} = ea_0n^2\left(T_1^1\boldsymbol{e}_1^* + T_0^1\boldsymbol{e}_0^* + T_{-1}^1\boldsymbol{e}_{-1}^*\right), \tag{14}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_0, \boldsymbol{e}_{-1}\}$ is the spherical basis, whose elements are given by

$$\boldsymbol{e}_1 = -\frac{1}{\sqrt{2}}\left(\boldsymbol{e}_x + i\boldsymbol{e}_y\right) \tag{15}$$

$$\boldsymbol{e}_0 = \boldsymbol{e}_z \tag{16}$$

$$\boldsymbol{e}_{-1} = \frac{1}{\sqrt{2}}\left(\boldsymbol{e}_x - i\boldsymbol{e}_y\right). \tag{17}$$

It can be recognized that to account the odd parity $\{T_{-1}^1, T_0^1, T_1^1\}$ should be considered as the *normalization-modified* spherical harmonics $\{\sqrt{4\pi}Y_{1-1}(\theta,\phi), \sqrt{4\pi}Y_{10}(\theta,\phi), \sqrt{4\pi}Y_{11}(\theta,\phi)\}$ [2].

C. Cavity QED and circuit QED

1. Electric dipole Hamiltonian

Now let us move on to investigate the interaction between the two level systems and electromagnetic environment. The dynamics of an electron in Bohr's atom in an electromagnetic environment can be formal described by the *minimal-coupling Hamiltonian* [3, 4]

$$H = \frac{1}{2m_e}\left[\boldsymbol{p} + e\boldsymbol{A}(\boldsymbol{r})\right]^2 + U(r) + H_R. \tag{18}$$

Under the assumption that an electron confined within a volume far smaller than the wavelength of the field (*long-wavelength approximation*) we can get more user-friendly *electric-dipole Hamiltonian*,

$$H' = \underbrace{\frac{\boldsymbol{p}^2}{2m_e} + U(r)}_{\text{Atomic part: } H_a} + \underbrace{H_{dip}}_{\text{Dipole selfenergy}} + \underbrace{\left(-\boldsymbol{d}\cdot\frac{\boldsymbol{D}'(0)}{\epsilon_0}\right)}_{\text{Interaction part: } H_{el}} + \underbrace{H_R}_{\text{Free field part}}, \tag{19}$$

from the minimal-coupling Hamiltonian Eq. (18) by performing a canonical transformation, the so-called *Power-Zienau-Woolley* transformation [3, 4]. Here $\boldsymbol{d}$ is the electric dipole moment defined by Eq. (14) for Bohr's atom. The *displacement* $\boldsymbol{D}'(r)$ after the transformation is related to the *transverse* electric field $\boldsymbol{E}_\perp(\boldsymbol{r})$ before the transformation as

$$\frac{\boldsymbol{D}'(r)}{\epsilon_0} = \boldsymbol{E}_\perp(\boldsymbol{r}). \tag{20}$$

Thus the interaction Hamiltonian represents the electric-dipole interaction

$$H_{el} = -\boldsymbol{d}\cdot\boldsymbol{E}_\perp(0). \tag{21}$$

Note that the atomic part H_a and the dipole self energy H_{dip} degenerate into Eq. (12) within the two level approximation.

More heuristic approach to the electric dipole interaction Hamiltonian Eq. (21) goes as follow [1]. The starting point is again the minimal-coupling Hamiltonian Eq. (18). By expanding the first term we have

$$H = \frac{\boldsymbol{p}^2}{2m_e} + \frac{e}{2m_e}\left(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}\right) + \frac{e^2}{2m_e}\boldsymbol{A}^2 + U(r) + H_R. \tag{22}$$

Under the weak filed condition $|\boldsymbol{p}| \gg |e\boldsymbol{A}|$ the third term $\frac{e^2}{2m_e}\boldsymbol{A}^2$ can be neglected. The second term can be simplified to

$$\frac{e}{2m_e}\left(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}\right) = \frac{e}{m_e}\boldsymbol{p}\cdot\boldsymbol{A} \tag{23}$$

by noticing the fact that

$$\begin{aligned}
\left[\boldsymbol{p}\cdot\boldsymbol{A}(\boldsymbol{r})\right]\psi(\boldsymbol{r}) &= -i\hbar\nabla\cdot\left(\boldsymbol{A}(\boldsymbol{r})\psi(\boldsymbol{r})\right) \\
&= -i\hbar\left(\underbrace{\nabla\cdot\boldsymbol{A}(\boldsymbol{r})}_{0\text{: Coulomb gauge}}\right)\psi(\boldsymbol{r}) + -i\hbar\boldsymbol{A}(\boldsymbol{r})\cdot\left(\nabla\psi(\boldsymbol{r})\right) \\
&= \left[\boldsymbol{A}(\boldsymbol{r})\cdot -i\hbar\nabla\right]\psi(\boldsymbol{r}) \\
&= \left[\boldsymbol{A}(\boldsymbol{r})\cdot\boldsymbol{p}\right]\psi(\boldsymbol{r}),
\end{aligned} \tag{24}$$

that is, $\boldsymbol{p}$ and $\boldsymbol{A}$ commute ($\boldsymbol{p}\cdot\boldsymbol{A} = \boldsymbol{A}\cdot\boldsymbol{p}$). Substituting

$$\boldsymbol{p} = m_e\frac{d\boldsymbol{r}}{dt} = m_e\frac{1}{i\hbar}\left[\boldsymbol{r}, H_a\right] \tag{25}$$

into Eq. (23) the transition probability from the upper state $|1\rangle$ to lower state $|0\rangle$ becomes

$$\begin{aligned}
\langle 0|\frac{e}{m_e}\boldsymbol{p}\cdot\boldsymbol{A}|1\rangle &= \frac{e}{i\hbar}\langle 0|\,[\boldsymbol{r},H_a]\cdot\boldsymbol{A}|1\rangle \\
&= \frac{e}{i\hbar}\langle 0|\,(\boldsymbol{r}H_a - H_a\boldsymbol{r})\,|1\rangle\cdot\boldsymbol{A} \\
&= \frac{e}{i\hbar}\,(\hbar\omega_1-\hbar\omega_0)\,\langle 0|\boldsymbol{r}|1\rangle\cdot\boldsymbol{A} \\
&= \underbrace{e\langle 0|\boldsymbol{r}|1\rangle}_{\langle 0|\boldsymbol{d}|1\rangle}\cdot\underbrace{(-i\omega_A)\boldsymbol{A}}_{\dot{\boldsymbol{A}}=-\boldsymbol{E}_\perp} \\
&= \langle 0|-\boldsymbol{d}\cdot\boldsymbol{E}_\perp|1\rangle. \qquad (26)
\end{aligned}$$

Thus the second term $\frac{e}{m_e}\boldsymbol{p}\cdot\boldsymbol{A}$ in Eq. (22) is related to the the electric dipole Hamiltonian H_{el}.

2. *Einstein's A coefficient*

Now that we have the interaction Hamiltonian H_{el} for the two level system coupled to the electromagnetic environment, we can see how the vacuum fluctuation or zero-point-fluctuation of the electromagnetic field causes the energy decay from the upper state $|1\rangle$ of the two level system to the lower state $|0\rangle$. The decay rate Γ_A from excited state to ground state is called Einstein's A coefficient, which can be calculated using Fermi's golden rule. Suppose initially that the polarization of the electric field is along x-axis:

$$\boldsymbol{E}_\perp(0,t) = i\sum_k\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\boldsymbol{e}_x\left(\hat{a}_k e^{-i\omega_k t}-\hat{a}_k^\dagger e^{i\omega_k t}\right). \qquad (27)$$

Then the relevant dipole moment which would interacts with the electric field Eq. (27) is that along $\boldsymbol{e}_x$, that is,

$$\boldsymbol{d} = \sqrt{2}\mu_n\hat{\sigma}_x\boldsymbol{e}_x = \mu_n\left(\frac{\hat{\sigma}_+}{\sqrt{2}}+\frac{\hat{\sigma}_-}{\sqrt{2}}\right)\boldsymbol{e}_x = \mu_n\left(-T_1^1+T_{-1}^1\right)\boldsymbol{e}_x, \qquad (28)$$

where $\mu_n = \frac{ea_0n^2}{\sqrt{2}}$ is the electric dipole moment of the transition $n\to n-1$ [5]. The interaction-picture interaction Hamiltonian can then be given by

$$V_I(t) = -i\sum_k\frac{\mu_n}{\sqrt{2}}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}\left(\hat{\sigma}_+e^{i\omega_A t}+\hat{\sigma}_-e^{-i\omega_A t}\right)\left(\hat{a}_k e^{-i\omega_k t}-\hat{a}_k^\dagger e^{i\omega_k t}\right), \qquad (29)$$

where the following relations are used:

$$\dot{\hat{\sigma}}_+ = \frac{1}{i\hbar}\left[\hat{\sigma}_+,H_a\right] = i\omega_A\hat{\sigma}_+ \;\Rightarrow\; \hat{\sigma}_+(t)=\hat{\sigma}_+(0)e^{i\omega_A t} \qquad (30)$$

$$\dot{\hat{\sigma}}_- = \frac{1}{i\hbar}\left[\hat{\sigma}_-,H_a\right] = -i\omega_A\hat{\sigma}_- \;\Rightarrow\; \hat{\sigma}_-(t)=\hat{\sigma}_-(0)e^{-i\omega_A t}, \qquad (31)$$

where H_a is the atomic Hamiltonian Eq. (12). Now invoking the *rotating-wave approximation*, the interaction Hamiltonian Eq. (29) is further simplified to be

$$\begin{aligned}
V_I(t) &= -i\sum_k\underbrace{\frac{\mu_n}{\sqrt{2}}\sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}}_{\lambda_k\hbar}\left(\hat{\sigma}_+\hat{a}_k e^{i(\omega_A-\omega_k)t}-\hat{\sigma}_-\hat{a}_k^\dagger e^{-i(\omega_A-\omega_k)t}\right) \\
&= -i\hbar\sum_k\lambda_k\left(\hat{\sigma}_+\hat{a}_k e^{i(\omega_A-\omega_k)t}-\hat{\sigma}_-\hat{a}_k^\dagger e^{-i(\omega_A-\omega_k)t}\right). \qquad (32)
\end{aligned}$$

The transition probability amplitude,

$$\alpha_{1\to 0}(t) = \langle 0|\hat{U_I}(t)|1\rangle, \qquad (33)$$

can be calculated by plugging the Dyson series

$$\hat{U}_I(t) = 1 - \frac{i}{\hbar}\int_0^t d\tau V_I(\tau)\left(1 - \frac{i}{\hbar}\int_0^t d\tau' V_I(\tau')\left(1 - \frac{i}{\hbar}\int_0^t d\tau'' V_I(\tau'')\cdots\right)\right) \approx 1 - \frac{i}{\hbar}\int_0^t d\tau V_I(\tau). \tag{34}$$

into the time-evolution operator $U_I(t)$ in Eq. (33), that is,

$$\begin{aligned}
\alpha_{1\to 0}(t) &= \langle 0|\hat{U}_I(t)|1\rangle \\
&= \underbrace{\langle 0|1\rangle}_{0} - \frac{i}{\hbar}\int_0^t d\tau \langle 0|\hat{V}_I(\tau)|1\rangle \\
&= -\int_0^t d\tau \sum_k \lambda_k \left(\underbrace{\langle 0|\hat{\sigma}_+|1\rangle}_{0}\hat{a}_k e^{i(\omega_A-\omega_k)\tau} - \underbrace{\langle 0|\hat{\sigma}_-|1\rangle}_{W_{10}}\hat{a}_k^\dagger e^{-i(\omega_A-\omega_k)\tau}\right) \\
&= \int_0^t d\tau \sum_k \lambda_k W_{10}\hat{a}_k^\dagger e^{-i(\omega_A-\omega_k)\tau}.
\end{aligned} \tag{35}$$

For *vacuum* we have the transition probability:

$$\begin{aligned}
P_{1\to 0}(t) &= \left\langle |\alpha_{1\to 0}(t)|^2 \right\rangle_0 \\
&= \int_0^t d\tau \int_0^t d\tau' \left\langle \left(\sum_k \lambda_k W_{10}\hat{a}_k e^{i(\omega_A-\omega_k)\tau}\right)\left(\sum_{k'} \lambda_{k'} W_{10}^* \hat{a}_{k'}^\dagger e^{-i(\omega_A-\omega_k)\tau'}\right)\right\rangle_0 \\
&= \int_0^t d\tau \int_0^t d\tau' \sum_k |\lambda_k W_{10}|^2 \underbrace{\langle \hat{a}_k \hat{a}_k^\dagger\rangle_0}_{1} e^{i(\omega_A-\omega_k)(\tau-\tau')} \\
&= \sum_k |\lambda_k W_{10}|^2 \int_0^t d\tau \underbrace{\int_0^t d\tau' e^{i(\omega_A-\omega_k)(\tau-\tau')}}_{2\pi\delta(\omega_A-\omega_k)} \\
&= 2\pi t \sum_k \delta\left(\omega_A - \omega_k\right) |\lambda_k W_{10}|^2 \\
&= 2\pi t \sum_k \delta\left(\omega_A - \omega_k\right) \frac{\mu_n^2}{2}\frac{\omega_k}{2\hbar\epsilon_0 V}|W_{10}|^2
\end{aligned} \tag{36}$$

The number of modes per unit volume can be obtained by the following geometric argument. Suppose the spherical shell of radius k and thickness dk in the reciprocal space. Since there are 2 polarization modes in unit cell $\left(\frac{2\pi}{L}\right)^3$ in the reciprocal space associated with the real space of volume $V = L^3$ the following relation holds:

$$2 : \left(\frac{2\pi}{L}\right)^3 = V\rho(k)dk : 4\pi k^2 dk, \tag{37}$$

where $\rho(k)$ is the *density of state* and we have

$$\rho(k)dk = \frac{k^2}{\pi^2}dk \tag{38}$$

The density of state with respect to ω can then be given by

$$\rho(\omega)d\omega = \frac{\omega^2}{\pi^2 c^2}d\omega. \tag{39}$$

Consequently the sum over k in Eq. (36) can be replaced by the integral over ω as

$$\sum_k \delta\left(\omega_A - \omega_k\right) \to \int_0^\infty \rho(k)dk\delta\left(\omega_A - \omega_k\right) = \int_0^\infty \rho(\omega)d\omega\delta\left(\omega_A - \omega\right) = \frac{\omega_A^2}{\pi^2 c^3}. \tag{40}$$

Note that this procedure of changing from sum to integral *implicitly* involved the *second continuum limit* $V \to \infty$ with

$$\hat{a}(\omega) = \frac{\hat{a}(k)}{\sqrt{c}} = \lim_{V\to\infty} \frac{\sqrt{V}\hat{a}_k}{\sqrt{c}}, \tag{41}$$

which satisfies the commutation relation

$$\left[\hat{a}(\omega), \hat{a}^\dagger(\omega')\right] = \left[\frac{\hat{a}(k)}{\sqrt{c}}, \frac{\hat{a}^\dagger(k')}{\sqrt{c}}\right] = \frac{2\pi}{c}\delta(k-k') = 2\pi\delta(\omega-\omega'). \tag{42}$$

The transition probability Eq. (36) becomes

$$\begin{aligned} P_{1\to 0}(t) &= 2\pi t \frac{\omega_A^2}{\pi^2 c^2}\frac{V}{c}\frac{\mu_n^2}{2}|W_{10}|^2 \frac{\omega_A}{2\hbar\epsilon_0 V} \\ &= t\ \frac{\mu_n^2}{2}|W_{10}|^2 \frac{\omega_A^3}{\pi\epsilon_0\hbar c^3}. \end{aligned} \tag{43}$$

The decay rate is thus

$$\Gamma_A = \frac{dP_{1\to 0}(t)}{dt} = \frac{\mu_n^2}{2}|W_{10}|^2 \frac{\omega_A^3}{\pi\epsilon_0\hbar c^3}. \tag{44}$$

Let us look at the part

$$\frac{\mu_n^2}{2}|W_{10}|^2 = \frac{\mu_n^2}{2}|\langle 0|\hat{\sigma}_-|1\rangle|^2 \tag{45}$$

more carefully. In terms of the irreducible tensor representation in Eq. (28) it can be rewritten in more informative form as

$$\frac{\mu_n^2}{2}|W_{10}|^2 = \frac{\mu_n^2}{2}|\langle 0|\sqrt{2}T^1_{-1}|1\rangle|^2 = \frac{1}{2J'+1}\sum_{M_J}\sum_{M_J'} \mu_n^2 |\langle g, J, M_J|T^1_{-1}|e, J', M_J'\rangle|^2 \tag{46}$$

where the initial state is averaged over the excited state sub-levels and the final state is the sum over the possible ground state sub-levels;

$$|1\rangle = \frac{1}{2J'+1}\sum_{M_J}|e, J', M_J'\rangle \tag{47}$$

$$|0\rangle = \sum_{M_J'}|g, J, M_J\rangle. \tag{48}$$

From the *Wigner-Eckart theorem* [1], the matrix element in Eq. (46) can be rewritten in terms of the *Clebsch-Gordan coefficient* as

$$|\langle g, J, M_J|T^1_{-1}|e, J', M_J'\rangle|^2 = \frac{|\langle g, J||T^1||e, J'\rangle|^2}{2J+1}|\langle J', M_J'; 1, -1|J, M_J\rangle|^2. \tag{49}$$

where $\langle g, J||T^1||e, J'\rangle$ is called the *reduced matrix element* of the tensor operator T^1, which is independent on the geometry of the system (independent on magnetic sub-levels M_J and M_J'). Using the following identity for the Clebsch-Gordan coefficient,

$$\begin{aligned} \sum_q\sum_{M_J}\sum_{M_J'}|\langle J', M_J'; 1, q|J, M_J\rangle|^2 &= \sum_{M_J}\left(\sum_{M_J'}\sum_q|\langle J', M_J'; 1, q|J, M_J\rangle|^2\right) \\ &= \sum_{M_J} 1 = 2J+1, \end{aligned} \tag{50}$$

we have

$$\sum_{M_J}\sum_{M_J'}|\langle J', M_J'; 1, -1|J, M_J\rangle|^2 = \frac{2J+1}{3}, \tag{51}$$

since the concerned space is isotropic the equality in choosing particular q out of three possible value $\{-1,0,1\}$ results. Plugging Eqs. (49) and (51) in Eq. (46) we have

$$\begin{aligned}\frac{\mu_n^2}{2}|W_{10}|^2 &= \frac{1}{2J'+1}\mu_n^2\frac{|\langle g,J||T^1||e,J'\rangle|^2}{2J+1}\left(\frac{2J+1}{3}\right)\\ &= \frac{1}{3}\mu_n^2\underbrace{\frac{|\langle g,J||T^1||e,J'\rangle|^2}{2J'+1}}_{1\ :\ \text{for}\ J'=J+1}\\ &= \frac{1}{3}\mu_n^2 = \frac{1}{3}\left(\frac{ea_0n^2}{\sqrt{2}}\right)^2. \end{aligned} \tag{52}$$

Plugging Eqs. (52) into the form of the decay rate Eq. (44) we have the famous Einstein A coefficient:

$$\Gamma_A = \frac{\mu_n^2\omega_A^3}{3\pi\epsilon_0\hbar c^3}. \tag{53}$$

The Einstein A coefficient Eq. (53) tells us (1) the decay rate increases third power in ω_A, which indicates the general trend that the two level systems with wider energy gap decay faster than those with narrow gap (2) the larger the dipole moment μ is the faster the decay rate becomes.

In this juncture let me introduce the *trivia* regarding the Einstein A coefficient Γ_A. First, the angular frequency ω_A of Bohr's atom for quantum number of $n=1$ (the orbit radius of $r_1=a_0$) becomes

$$\omega_A = \frac{\hbar}{m_e r_1^2} = \frac{\hbar}{m_e a_0^2} \tag{54}$$

thus

$$\mu_1^2 = \left(\frac{ea_0}{\sqrt{2}}\right)^2 = \left(e\underbrace{\sqrt{\frac{\hbar}{2m_e\omega_A}}}_{x_{\text{zpf}}}\right)^2 = \left(ex_{\text{zpf}}\right)^2. \tag{55}$$

where μ_1 is the dipole moment of *classical electron oscillator*, an electron in a harmonic trap with the trap angular frequency of ω_A and the its r.m.s. displacement is zero-point-fluctuation x_{zpf}. The radiative decay rate of the classical electron oscillator can then be obtained by plugging μ_1 of Eq. (55) into μ_n in Eq. (53):

$$\Gamma_1 = \frac{\mu_1^2\omega_A^3}{3\pi\epsilon_0\hbar c^3} = \frac{e^2\omega_A^2}{6\pi\epsilon_0 m_e c^3}, \tag{56}$$

which does not contain $\hbar$.

Second, the dimensionless *oscillator strength* $\mathcal{F}_{ij}$ of the transition from level j down to level i characterized by γ_{ij} [6] can be defined as

$$\mathcal{F}_{ij} = \frac{\gamma_{ij}}{3\Gamma_1}, \tag{57}$$

where the factor 3 in the denominator is to undo the average we have performed in Eq. (51). This quantifies the decay rate of an atom (or an artificial atom) by comparing it with that of the classical electron oscillator [6].

Finally, it is instructive to see the relation between the Einstein A coefficient and the *Larmor formula* of radiation power P_r emitted from the classical dipole, [5]:

$$\begin{aligned}\hbar\omega_A\Gamma_A &= \hbar\omega_A\frac{\mu_n^2\omega_A^3}{3\pi\epsilon_0\hbar c^3} = \frac{\left(\frac{ea_0n^2}{\sqrt{2}}\right)^2\omega_A^4}{3\pi\epsilon_0 c^3} = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\omega_A^4a_0^2n^4\right)\\ &= \frac{e^2}{6\pi\epsilon_0 c^3}\left(\omega_A^2 r_n\right)^2 = \frac{e^2}{6\pi\epsilon_0 c^3}\left(\ddot{r}_n\right)^2 = P_r. \end{aligned} \tag{58}$$

3. Purcell factor

The number of modes within the bandwidth between $\omega + d\omega$ and ω for free space is:

$$N_A(\omega) = \frac{V\omega^2}{\pi^2 c^3}, \tag{59}$$

while that for cavity

$$N_c(\omega) = \frac{\kappa}{2\pi} \frac{1}{\left((\omega - \omega_c)^2 + \left(\frac{\kappa}{2}\right)^2\right.}, \tag{60}$$

The free-space spontaneous emission rate,

$$\Gamma_A(\omega) = 2\pi \ \frac{V\omega^2}{\pi^2 c^3} \ \frac{\mu^2}{3} \ \frac{\omega}{2\hbar\epsilon_0 V}, \tag{61}$$

can be modified if we place the cavity with quality factor of $Q = \frac{\omega}{\kappa}$ and volume of V as

$$\Gamma_c(\omega_c) = 2\pi \ \frac{2}{\pi} \frac{Q}{\omega} \ \mu^2 \ \frac{\omega}{2\hbar\epsilon_0 V}, \tag{62}$$

We have thus the Purcell factor

$$F_{\mathrm{P}} = \frac{\Gamma_c}{\Gamma_A} = \frac{3}{4\pi^2} \frac{Q}{V} \lambda^3. \tag{63}$$

[1] D. Budker, D. F. J. Kimball, and D. DeMille, *Atomic Physics*, 2nd ed. (Oxford University Press, Oxford, England, 2008).
[2] C. J. Foot, *Atomic Physics* (Oxford University Press, Oxford, 2005).
[3] R. Loudon, *The Quantum Theory of Light*, 3rd ed. (Oxford University Press, New York, 2000).
[4] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics*, (Wiley, New York, 1997).
[5] M. H. Devoret, S. Girvin, and R. Schoelkopf, Ann. Phys. (Leipzig) **16,** 767 (2007).
[6] A. E. Siegman, *Lasers*, (University Science, Mill Valley, CA, 1986).