Quantization of environment

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We have learned harmonic oscillators, coupled harmonic oscillators, and Bosonic fields, which are emerged as the contiuum limit ($N \to \infty$) of the N coupled harmonics oscillators. With these ingredients we venture into the environmental problem of quantum mechanics. In particular we study an LCR circuit as an example. We learn how the enegy dissipation of LC circuit (the *system*) due to the resistor R (the *environment*) can be understood quantum mechanically by treating R as a semi-infinite 1D transmission line with characteristic impedance of $Z_p = R$. The model that the dissipative elements are treated as a collection of conservative (reactive) elements is called the *Caldeira-Leggett model*. The environment which is characterized by the *frequency-independent* impedance Z_p is called *Ohmic environment*.

I. LCR CIRCUIT - A CLASSICAL VIEW

Let us start by discussing the LCR circuit classically. The circuit equation is given by the Langevin-type equation:

$$L_0\ddot{Q}(t) + R\dot{Q}(t) + \frac{1}{C_0}Q(t) = \underbrace{\mathcal{V}(t)}_{\text{Noise voltage}} . \tag{1}$$

The Fourier transform:

$$Q(t) = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} Q(\Omega) e^{-i\Omega t} \tag{2}$$

$$Q(\Omega) = \int_{-\infty}^{\infty} dt Q(\Omega) e^{i\Omega t} \tag{3}$$

gives

$$-\Omega^2 L_0 Q(\Omega) - iR\Omega Q(\Omega) + \frac{1}{C_0}Q(\Omega) = \mathcal{V}(\Omega). \tag{4}$$

Putting $\Omega_0 = \sqrt{\frac{1}{L_0 C_0}}$ we have the following algebraic equation:

$$\begin{aligned} \mathcal{V}(\Omega) &= \left[L_0\left(\Omega_0 - \Omega^2\right) - i\Omega R\right] Q(\Omega) \\ &= \chi_v(\Omega) Q(\Omega). \end{aligned} \tag{5}$$

Here the susceptibility $\chi_v(\Omega)$ is intoduced. $\chi_v(\Omega)$ can be split into in-phase and quadarure parts, that is, $\chi_v(\Omega) = \chi_v'(\Omega) + i\chi_v''(\Omega)$ with

$$\chi_v'(\Omega) = L_0\left(\Omega_0^2 - \Omega^2\right) \tag{6}$$

$$\chi_v''(\Omega) = -\Omega R. \tag{7}$$

We can thus think that the noise voltage $\mathcal{V}(t)$ at the resistor is induced as the response of the motion of the charge $Q(t)$.

From Eq. (5) we get the spectral density of the charge $\bar{S}_{QQ}(\Omega)$ as

$$\begin{aligned} \bar{S}_{QQ}(\Omega) = Q^*(\Omega)Q(\Omega) &= \left|\frac{1}{\chi_v(\Omega)}\right|^2 \mathcal{V}^*(\Omega)\mathcal{V}(\Omega) \\ &= \left|\frac{1}{\chi_v(\Omega)}\right|^2 \bar{S}_{VV}(\Omega), \end{aligned} \tag{8}$$

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where $\bar{S}_{VV}(\Omega)$ is the spectral density of the voltage. Here $\bar{S}_{QQ}(\Omega)$ and $\bar{S}_{VV}(\Omega)$ are the *single-sided spectral densities*, which only assumes positive frequency ($\Omega \geq 0$). The variance of $\langle Q(t)^2\rangle$ is obtained by integrating Eq. (8). To see how this connection arises let us invoke the *Wiener-Khinchin therom*:

$$\begin{aligned}\langle Q(\tau)Q(0)\rangle &= \int_0^\infty \frac{d\Omega}{2\pi}\bar{S}_{QQ}(\Omega)\cos\Omega\tau \qquad (9)\\ \bar{S}_{QQ}(\Omega) &= \int_{-\infty}^\infty d\tau\langle Q(\tau)Q(0)\rangle\cos\Omega\tau, \qquad (10)\end{aligned}$$

Plugging $\tau = 0$ in Eq. (11) gives us the variance of $Q(t)$, that is,

$$\begin{aligned}\langle Q(t)^2\rangle \underbrace{=}_{\text{Stationarity}} \langle Q(0)^2\rangle &= \int_0^\infty \frac{d\Omega}{2\pi}\bar{S}_{QQ}(\Omega)\\ &= \int_0^\infty \frac{d\Omega}{2\pi}\left|\frac{1}{\chi_v(\Omega)}\right|^2\bar{S}_{VV}(\Omega)\\ &= \int_0^\infty \frac{d\Omega}{2\pi}\frac{1}{L_0^2}\left(\frac{1}{\left(\Omega_0^2-\Omega^2\right)^2+\left(\frac{R}{L_0}\right)^2\Omega^2}\right)\bar{S}_{VV}(\Omega) \qquad (11)\end{aligned}$$

When we are interested in the system in the thermal equilibrium the variance $\langle Q(t)^2\rangle$ can be obtained from the thermodynamical reasoning. Invoking the *equipartition therem* we have

$$\frac{1}{2}\frac{\langle Q(t)^2\rangle}{C_0} = \frac{1}{2}k_\mathrm{B}T. \qquad (12)$$

Now suppose that the spectral density of voltage is white, that is, $\bar{S}_{VV}(\Omega) = \bar{S}_{VV}$. By performing the integration in Eq. (11) we have

$$\begin{aligned}\langle Q(t)^2\rangle &= \frac{\bar{S}_{VV}}{L_0^2}\int_0^\infty \frac{d\Omega}{2\pi}\left(\frac{1}{\left(\Omega_0^2-\Omega^2\right)^2+\left(\frac{R}{L_0}\right)^2\Omega^2}\right)\\ &= \frac{1}{4L_0\Omega_0R}\bar{S}_{VV}. \qquad (13)\end{aligned}$$

By comparing Eqs. (12) and (13) we arrived at the well-known *Nyquist formula*:

$$\bar{S}_{VV} = 4Rk_\mathrm{B}T, \qquad (14)$$

which shows the connection between the Ohmic dissipation R, the voltage fluctuation $\bar{S}_{VV}$, and the temperature. This can be recast into the following form

$$R = \frac{1}{2k_\mathrm{B}T}S_{VV}(\Omega), \qquad (15)$$

where we use the *double-sided* spectral density $S_{VV}(\Omega)$, which assumes potive and negative frequency. The single-sided spectral density $\bar{S}_{VV}(\Omega)$ can then be obtained in terms of them as

$$\bar{S}_{VV}(\Omega) = S_{VV}(\Omega) + S_{VV}(-\Omega). \qquad (16)$$

In classical setting we have $S_{VV}(\Omega) = S_{VV}(-\Omega)$, which can be derived from the fact that $\langle V(t)V(0)\rangle = \langle V(0)V(t)\rangle$, that is, $V(t)$ and $V(0)$ commute.

II. QUANTIZING THE ENVIRONMENT [1, 2]

A. Hamiltonian formalism

Let us reexamine the LCR circuit from the viewpoint of Hamiltonian formalism hoping that we will gain more general tools to tackle open quantum systems. After taking the first and second continuum limits the Hamiltonian of a 1D transmission line can be expressed as

$$H_0 = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar\omega_k \left(\hat{c}^\dagger(k)\hat{c}(k) + \frac{1}{2} \right), \tag{17}$$

where

$$\hat{c}(k) = \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(-k) + \frac{i}{c\omega_k} q(k) \right) \tag{18}$$

$$\hat{c}^\dagger(k) = \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(k) - \frac{i}{c\omega_k} q(-k) \right), \tag{19}$$

are the annihilation and creation operators with the commutation relation

$$[\hat{c}(k), \hat{c}^\dagger(k)] = 2\pi\delta(k - k'). \tag{20}$$

Here the flux operator $\varphi(k)$ and the charge operator $q(k)$ are given by

$$\varphi(k) = = \int_{-\infty}^{\infty} dx \varphi(x) e^{-ikx} \tag{21}$$

$$q(k) = \int_{-\infty}^{\infty} dx q(x) e^{ikx}, \tag{22}$$

respectively with the commutation relation:

$$[\varphi(k), q(k')] = i\hbar \; 2\pi\delta(k - k'). \tag{23}$$

The Heisenberg equations of motion for $\hat{c}(k)$ and $\hat{c}^\dagger(k)$ are

$$\dot{\hat{c}}(k,t) = \frac{i}{\hbar} [H_0, \hat{c}(k,t)] = -i\omega_k \hat{c}(k,t) \tag{24}$$

$$\dot{\hat{c}}^\dagger(k,t) = \frac{i}{\hbar} \left[H_0, \hat{c}^\dagger(k,t)\right] = i\omega_k \hat{c}^\dagger(k,t) \tag{25}$$

thus we have the plane wave solutions:

$$\hat{c}(k,t) = \hat{c}(k,0) e^{-i\omega_k t} \tag{26}$$

$$\hat{c}^\dagger(k,t) = \hat{c}^\dagger(k,0) e^{i\omega_k t}. \tag{27}$$

With these results the charge variable $q(x,t)$ is given by

$$\begin{aligned} q(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} q(k,t) e^{ikx} \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} i\sqrt{\frac{\hbar\omega_k c}{2}} \left(\hat{c}^\dagger(-k,t) - \hat{c}(k,t) \right) e^{ikx} \\ &= -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar\omega_k c}{2}} \left(\hat{c}(k,0) e^{i(kx - \omega_k t)} - h.c. \right), \end{aligned} \tag{28}$$

which is indeed manifestly real as it has to be. The voltage V(x,t), which is also a real quantity, can be written in terms of $q(x,t)$ as

$$V(x,t) = \frac{q(x,t)}{c} = -i \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar\omega_k}{2c}} \left(\hat{c}(k,0) e^{i(kx-\omega_k t)} - h.c. \right). \tag{29}$$

We now identify the modes with positive k as the right-moving modes and those with negative k as the left-moving modes. The right-moving voltage $V^{\rightarrow}(x,t)$ can thus be given by

$$\begin{aligned} V^{\rightarrow}(x,t) &= -i \int_0^{\infty} \frac{dk}{2\pi} \sqrt{\frac{\hbar\omega_k}{2c}} \left(\hat{c}(k,0) e^{i(kx-\omega_k t)} - h.c. \right) \\ &= -i \int_0^{\infty} \frac{v_p dk}{2\pi} \sqrt{\frac{\hbar\omega_k}{2cv_p}} \left(\frac{\hat{c}(k,0)}{\sqrt{v_p}} e^{i(kx-\omega_k t)} - h.c. \right) \\ &= -i \int_0^{\infty} \frac{d\omega}{2\pi} \sqrt{\frac{\hbar\omega Z_p}{2}} \left(\hat{c}(\omega) e^{i(kx-\omega t)} - h.c. \right) \end{aligned} \tag{30}$$

and the left-moving voltage can similarly given by

$$V^{\leftarrow}(x,t) = -i\int_{-\infty}^{0}\frac{d\omega}{2\pi}\sqrt{\frac{\hbar\omega Z_p}{2}}\left(\hat{c}(\omega)e^{i(kx-\omega t)} - h.c.\right), \tag{31}$$

where $\hat{c}(\omega) = \frac{\hat{c}(k,0)}{\sqrt{v_p}}$, which satisfies the commutation relation:

$$[\hat{c}(\omega), \hat{c}^\dagger(\omega')] = [\frac{\hat{c}(k)}{\sqrt{v_p}}, \frac{\hat{c}^\dagger(k')}{\sqrt{v_p}}] = \frac{2\pi}{v_p}\delta(k-k') = 2\pi\delta(\omega-\omega'). \tag{32}$$

While the average voltage fluctuation $\langle V(x,t)\rangle_t$ is zero under the thermal equilibrium the variance is not, which is basically the *Johnson-Nyquist noise*. By evaluating the variance, or rather the spectral density $S_{VV}(\omega)$, we shall find the quantum version of the *Nyquist formula*. Let us consider the auto-correlation of the voltage at the open terminal at $x=0$ of a semi-infinite transmission line with the characteristic impedance $Z_p = \sqrt{\frac{l}{c}}$, which can be given by

$$\begin{aligned}
\langle V(0,t+\tau)V(0,t)\rangle_t &= \langle \left(V^{\rightarrow}(0,t+\tau) + V^{\leftarrow}(0,t+\tau)\right)\left(V^{\rightarrow}(0,t) + V^{\leftarrow}(0,t)\right)\rangle_t \\
&= 4\langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle_t,
\end{aligned} \tag{33}$$

where the stationarity leads to the first equation, and $V(x,t) = V^{\rightarrow}(x,t) + V^{\leftarrow}(x,t)$ and $V^{\rightarrow}(x,t) = V^{\leftarrow}(x,t)$ for the open terminal lead to the second and third equation, respectively.

For the situation in which the stationarity condition is satisfied the spectral density is obtained via the *Wiener-Khinchin theorem*:

$$\begin{aligned}
S_{VV}(\Omega) &= \int_{-\infty}^{\infty} d\tau \langle V(0,t+\tau)V(0,t)\rangle_t e^{i\Omega\tau} \\
&= 4\int_{-\infty}^{\infty} d\tau \langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle_t e^{i\Omega\tau} = 4S_{VV}^{\rightarrow}(\Omega).
\end{aligned} \tag{34}$$

With Eq. (30) we have

$$\begin{aligned}
S_{VV}^{\rightarrow}(\Omega) &= \int_{-\infty}^{\infty} d\tau \langle V^{\rightarrow}(0,t+\tau)V^{\rightarrow}(0,t)\rangle e^{i\Omega\tau} \\
&= -\int_{-\infty}^{\infty} d\tau \int_0^{\infty}\frac{d\omega'}{2\pi}\int_0^{\infty}\frac{d\omega'}{2\pi}\frac{\hbar Z_p}{2}\sqrt{\omega\omega'}\left(\underbrace{\langle\hat{c}(\omega)c(\omega')e^{-i(\omega+\omega')t}\rangle}_{0} - \underbrace{\langle\hat{c}(\omega)c^\dagger(\omega')e^{-i(\omega-\omega')t}\rangle}_{(n(\omega)+1)2\pi\delta(\omega-\omega')}\right)e^{i(\Omega-\omega)\tau} \\
&\qquad + \left(-\underbrace{\langle\hat{c}^\dagger(\omega)c(\omega')e^{-i(-\omega+\omega')t}\rangle}_{n(\omega)2\pi\delta(\omega-\omega')} + \underbrace{\langle\hat{c}^\dagger(\omega)c^\dagger(\omega')e^{-i(-\omega-\omega')t}\rangle}_{0}\right)e^{i(\Omega+\omega)\tau} \\
&= \int_{-\infty}^{\infty} d\tau \int_0^{\infty}\frac{d\omega}{2\pi}\frac{\hbar\omega Z_p}{2}\left(\left(n(\omega)+1\right)e^{i(\Omega-\omega)\tau} + n(\omega)e^{i(\Omega+\omega)\tau}\right) \\
&= \int_0^{\infty} d\omega\frac{\hbar\omega Z_p}{2}\left(\left(n(\omega)+1\right)\delta(\Omega-\omega) + n(\omega)\delta(\Omega+\omega)\right) \\
&= \frac{\hbar|\Omega|Z_p}{2}\left(\left(n(\Omega)+1\right)\Theta(\Omega) + n(|\Omega|)\Theta(-\Omega)\right),
\end{aligned} \tag{35}$$

where $\Theta(x)$ is the step function. Thus we have the voltage noise spectrum:

$$S_{VV}(\Omega) = 4S_{VV}^{\rightarrow}(\Omega) = 2\hbar|\Omega|Z_p\left(\left(n(\Omega)+1\right)\Theta(\Omega) + n(|\Omega|)\Theta(-\Omega)\right). \tag{36}$$

B. Anatomy of the Johnson-Nyquist noise

Let us now take a step back and see what is going on here. For the real-valued *classical* variable $V(\tau)$ its auto-correlation function $G_{VV}(\tau) = \langle V(\tau)V(0)\rangle$ is also real. The commutativity of classical variable also suggests $G_{VV}(\tau) =$

$G_{VV}(-\tau)$, that is, the auto-correlation is symmetric in time. This leads to the *symmetric-in-frequency* power spectrum:

$$\begin{aligned} S_{VV}(-\Omega) &= \int_{-\infty}^{\infty} d\tau G_{VV}(\tau) e^{-i\Omega\tau} \\ &= \int_{\infty}^{-\infty} (-d\tau) \underbrace{G_{VV}(-\tau)}_{G_{VV}(\tau)} e^{i\Omega\tau} = S_{VV}(\Omega). \end{aligned} \tag{37}$$

For the real-valued *quantum* variable $V(\tau)$, however, its auto-correlation function $G_{VV}(\tau)$ is not necessarily real! Let us see this in the following simple argument with a LC circuit. The real-valued flux variable is given by

$$\varphi(t) = \sqrt{\frac{\hbar}{2C_0\omega}} \left(\hat{c}(t) + \hat{c}^\dagger(t)\right) = \sqrt{\frac{\hbar Z_0}{2}} \left(\hat{c}e^{-i\omega_0 t} + \hat{c}^\dagger e^{i\omega_0 t}\right), \tag{38}$$

which is manifestly hermitian. The auto-correlation function is, however, *not* hermitian:

$$\begin{aligned} G_{\varphi\varphi}(\tau) &= \frac{\hbar Z_0}{2} \left(\langle \hat{c}\hat{c}^\dagger\rangle e^{-i\omega_0 t} + \langle \hat{c}^\dagger \hat{c}\rangle e^{i\omega_0 t}\right) \\ &= \frac{\hbar Z_0}{2} \left((n(\omega_0)+1) e^{-i\omega_0 t} + n(\omega_0) e^{i\omega_0 t}\right). \end{aligned} \tag{39}$$

Thus we arrive the *asymmetric-in-frequency* power spectrum power spectrum:

$$S_{\varphi\varphi}(\Omega) = \int_{-\infty}^{\infty} d\tau G_{\varphi\varphi}(\tau) e^{i\Omega\tau} = \frac{\hbar Z_0}{2} \left((n(\omega_0)+1) 2\pi\delta(\Omega-\omega_0) + n(\omega_0) 2\pi\delta(\Omega+\omega_0)\right). \tag{40}$$

Since $V(t) = \dot{\varphi}(t)$ by the similar argument we have the *asymmetric-in-frequency* power spectrum:

$$S_{VV}(\Omega) = \frac{\hbar\omega_0^2 Z_0}{2} \left((n(\omega_0)+1) 2\pi\delta(\Omega-\omega_0) + n(\omega_0) 2\pi\delta(\Omega+\omega_0)\right), \tag{41}$$

which is the discrete version of the spectral density in Eq. (36). We see that the non-commutativity of the quantum operators $\varphi(t)$ and $V(t)$ with those in different time and the emergent *quantum fluctuation* (i.e., *one extra photon* in the positive frequency) is the culprit of the *asymmetric-in-frequency* power spectrum power spectrum. We also see that the result we have in Eq. (36) can be obtained by adding the contribution of infinitely many LC circuits with different frequencies.

C. Quantum dissipation-fluctuation theorem

The expression Eq. (36) can recast into more compact form:

$$\begin{aligned} S_{VV}(\Omega) &= 2Z_p \left(\hbar\Omega \left(\frac{1}{e^{\frac{\hbar\Omega}{k_B T}} - 1} + 1 \right) \Theta(\Omega) - \hbar\Omega \left(\frac{1}{e^{-\frac{\hbar\Omega}{k_B T}} - 1} \right) \Theta(-\Omega) \right) \\ &= 2Z_p \left(\hbar\Omega \left(\frac{1}{1 - e^{-\frac{\hbar\Omega}{k_B T}}} \right) \Theta(\Omega) + \hbar\Omega \left(\frac{1}{1 - e^{-\frac{\hbar\Omega}{k_B T}}} \right) \Theta(-\Omega) \right) \\ &= \left(\frac{2Z_p\hbar\Omega}{1 - e^{-\frac{\hbar\Omega}{k_B T}}} \right). \end{aligned} \tag{42}$$

This is called a double-sided spectral density where the frequency Ω runs from negative to positive as shown in Fig. 1. Although the asymmetry of the spectral density in frequency is noticeable, it is not so trivial to see the asymmetry in practice. The reason is that the range of frequencies where the asymmetry is significant is $\Omega > \frac{k_B T}{\hbar}$, that is very high for the room temperature and GHz range even in a few mK environment. It is also required to distinguish between the positive and negative frequencies in order to see the asymmetry. Nevertheless there are several experiments in which shows the existence of quantum noise [3–8].

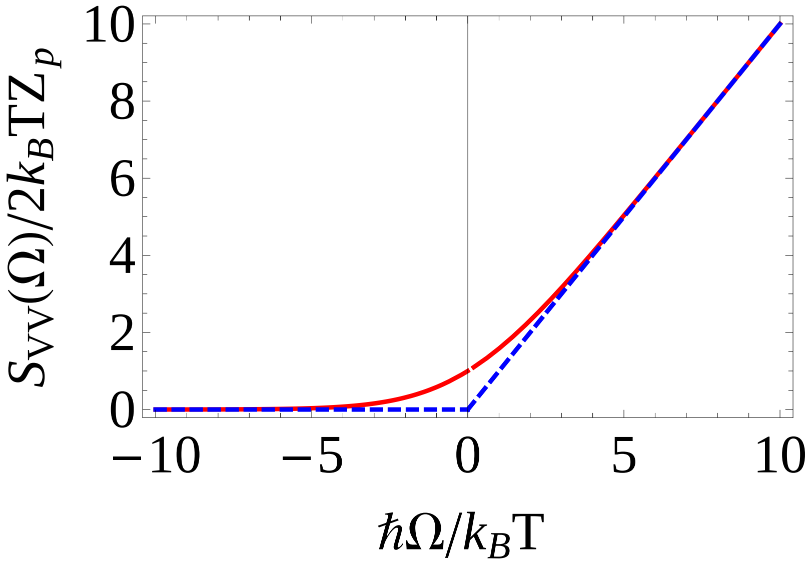

FIG. 1: Spectral density of quantum Johnson-Nyquist noise across a impedance matched to the characteristic impedance of the 1D transmission line, Z_p. The blue dashed line shows the zero-temperature quantum noise while the red line shows the one at finite temperature.

The single-sided spectral density, which can be measured with standard spectrum analyzers, is, on the other hand, given by symmetrizing the spectral density with respect to frequency:

$$\begin{aligned} \bar{S}_{VV}(\Omega) = S_{VV}(\Omega) + S_{VV}(-\Omega) &= \left(\frac{2Z_p\hbar\Omega}{1-e^{-\frac{\hbar\Omega}{k_\mathrm{B}T}}}\right) + \left(\frac{-2Z_p\hbar\Omega}{1-e^{\frac{\hbar\Omega}{k_\mathrm{B}T}}}\right) \\ &= 2Z_p\hbar\Omega\coth(\frac{\hbar\Omega}{2k_\mathrm{B}T}) \qquad (43) \\ &= 4Z_p\hbar\Omega\left\{\underbrace{\frac{1}{e^{\frac{\hbar\Omega}{k_\mathrm{B}T}}-1}}_{n(\hbar\Omega)}+\frac{1}{2}\right\}, \qquad (44) \end{aligned}$$

where the frequency Ω runs only in the positive direction. In the last line we can recognize the contribution of the *zero point fluctuation*, $2Z_p\hbar\Omega$, to the noise spectral density explicitly. Equation (44) is called *quantum dissipation-fluctuation theorem*, which connects the apparently unrelated two quantities; the transport coefficient Z_p and the noise spectral density $\bar{S}_{VV}(\Omega)$.

By taking the classical limit $k_\mathrm{B}T \gg \hbar\Omega$ the spectral density Eq. (44) becomes

$$\bar{S}_{VV}(\Omega) = 4Z_pk_\mathrm{B}T, \qquad (45)$$

which is the well-known *Johnson-Nyquist formula*, where the spectrum is proportional to the impedance Z_p and temperature T

The impedance Z_p is, on the other hand, related to the *difference* of the noise spectral densities $S_{VV}(\Omega)$ and $S_{VV}(-\Omega)$;

$$S_{VV}(\Omega) - S_{VV}(-\Omega) = 2Z_p\left(\frac{\hbar\Omega}{1-e^{-\frac{\hbar\Omega}{k_\mathrm{B}T}}} - \frac{-\hbar\Omega}{1-e^{\frac{\hbar\Omega}{k_\mathrm{B}T}}}\right) = 2Z_p\hbar\Omega, \qquad (46)$$

that is,

$$Z_p = \frac{1}{2\hbar\Omega}\left(S_{VV}(\Omega) - S_{VV}(-\Omega)\right). \qquad (47)$$

D. Ohmic environment

We are thus able to treat a dissipative element characterized by the impedance Z_p quantum mechanically. The quantum dissipation fluctuation theorem Eq. (44) shows peculiar quantum effect which manifest itself as the *asymmetric-in-frequency* power spectrum in the quantum regime $k_{\rm B}T \leq \hbar\Omega$. That the dissipative elements can be treated as a collection of conservative (reactive) elements is essentially the way in which the *Caldeira-Leggett model* deals with resisters quantum mechanically [1, 2]. The environment which is characterized by the *frequency-independent* impedance Z_p and has the noise power spectrum Eq. (42) is called *Ohmic environment.*

III. RELATION TO THE LINEAR-RESPONSE THEORY [2, 9, 10]

This kind of transport coefficient can in general be obtained by the *linear-response theory.* Let us check the above result can be reproduced from the linear-response theory. Suppose that the Hamiltonian H_0 in Eq. (17) describes the unperturbed Hamiltonian for a 1D transmission line, which is now connected at $x = 0$ to an LC circuit capacitively with the intercation Hamiltonian

$$H_i = \hat{Q}_s(t)\hat{V}, \tag{48}$$

where the canonical variables of the LC circuit are the charge $\hat{Q}_s(t)$ and the flux $\hat{\Phi}_s(t)$ and $\hat{V} = \hat{V}(x = 0)$ is Schrödinger's operator for the voltage at $x = 0$ (see Eq. (29)). In the interaction picture the time evolution of the dentiy operator for the 1D transmission line $\rho_I(t)$ is given by

$$\frac{\partial}{\partial t}\rho_I(t) = -\frac{i}{\hbar}\left[H_I(t), \rho_I(t)\right], \tag{49}$$

where

$$H_I(t) = e^{i\frac{H_0}{\hbar}t} H_i e^{-i\frac{H_0}{\hbar}t} = \hat{Q}_s(t)\left(e^{i\frac{H_0}{\hbar}t}\hat{V}e^{-i\frac{H_0}{\hbar}t}\right) = \hat{Q}_s(t)\hat{V}_I(t) \tag{50}$$

and

$$\rho_I(t) = e^{i\frac{H_0}{\hbar}t}\rho(t)e^{-i\frac{H_0}{\hbar}t} \tag{51}$$

with $\rho(t)$ denoting the Schrödinger's dentiy operator.

Assuming that the 1D transmission line is the initially unperturbed and in the thermal equilibrium state $\rho_{eq} = \rho(-\infty) = \rho_I(-\infty)$. Then the formal solution of Eq. (49) can be obtained perturbatively as

$$\begin{aligned}
\rho_I(t) &= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \left[H_I(t'), \rho_I(t')\right] \\
&= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \left[H_I(t'), \left(\rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{t'} dt'' \left[H_I(t''), \rho_I(t'')\right]\right)\right] \\
&\sim \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{t} dt' \left[H_I(t'), \rho_{eq}\right] \\
&= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\theta(t-t') \left[H_I(t'), \rho_{eq}\right] \\
&= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\theta(t-t') \left[\hat{V}_I(t'), \rho_{eq}\right],
\end{aligned} \tag{52}$$

where $\theta(t - t')$ is the step function:

$$\theta(t-t') = \begin{cases} 1, & \text{if } t - t' \geq 0 \\ 0, & \text{if } t - t' < 0, \end{cases} \tag{53}$$

used for extending the domain of integration to the infinity. Returning to the Schrödinger's density operator gives

$$\begin{aligned}
\rho(t) &= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\theta(t-t') \left[e^{i\frac{H_0}{\hbar}(t'-t)}\hat{V}e^{-i\frac{H_0}{\hbar}(t'-t)}, \rho_{eq}\right] \\
&= \rho_{eq} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\theta(t-t') \left[\hat{V}_I(t'-t), \rho_{eq}\right]
\end{aligned} \tag{54}$$

The expectation value of $\hat{V}$ is then written as

$$\begin{aligned}
\langle V(t)\rangle &= \mathrm{Tr}\left[\rho(t)\hat{V}\right] \\
&= \underbrace{\mathrm{Tr}\left[\rho_{eq}\hat{V}\right]}_{0} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\theta(t-t')\mathrm{Tr}\left[\left[\hat{V}_I(t'-t),\rho_{eq}\right]\hat{V}\right] \\
&= -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\theta(t-t')\mathrm{Tr}\left[\left(\hat{V}\hat{V}_I(t'-t)-\hat{V}_I(t'-t)\hat{V}\right)\rho_{eq}\right] \\
&= \int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\underbrace{\left(-\frac{i}{\hbar}\theta(t-t')\mathrm{Tr}\left[\left(\hat{V}\hat{V}_I(t'-t)-\hat{V}_I(t'-t)\hat{V}\right)\rho_{eq}\right]\right)}_{\chi_v(t-t')} \\
&= \int_{-\infty}^{\infty} dt'\hat{Q}_s(t')\chi_v(t-t'),
\end{aligned} \tag{55}$$

where we define the time-domain response function $\chi_v(\tau)$ as

$$\begin{aligned}
\chi_v(\tau) &= -\frac{i}{\hbar}\theta(\tau)\mathrm{Tr}\left[\left(\hat{V}\hat{V}_I(-\tau)-\hat{V}_I(-\tau)\hat{V}\right)\rho_{eq}\right] \\
&= -\frac{i}{\hbar}\theta(\tau)\left(\langle\hat{V}(0)\hat{V}(-\tau)\rangle-\langle\hat{V}(-\tau)\hat{V}(0)\rangle\right) \\
&= -\frac{i}{\hbar}\theta(\tau)\left(\langle\hat{V}(\tau)\hat{V}(0)\rangle-\langle\hat{V}(0)\hat{V}(\tau)\rangle\right).
\end{aligned} \tag{56}$$

The Fourier transform of Eq. (56) gives the susceptibility $\chi_v(\Omega)$:

$$\begin{aligned}
\chi_v(\Omega) &= \chi_v'(\Omega)+i\chi_v''(\Omega) \\
&= -\frac{i}{\hbar}\int_{-\infty}^{\infty} d\tau\theta(\tau)\left(\langle\hat{V}(\tau)\hat{V}(0)\rangle-\langle\hat{V}(0)\hat{V}(\tau)\rangle\right)e^{i\Omega\tau} \\
&= -\frac{i}{\hbar}\int_{0}^{\infty} d\tau\left(\langle\hat{V}(\tau)\hat{V}(0)\rangle-\langle\hat{V}(0)\hat{V}(\tau)\rangle\right)e^{i\Omega\tau}.
\end{aligned} \tag{57}$$

The imaginary part $\chi_v''(\Omega)$ can then read

$$\begin{aligned}
\chi_v''(\Omega) &= -\frac{1}{\hbar}\mathrm{Re}\left[\int_{0}^{\infty} d\tau\left(\langle\hat{V}(\tau)\hat{V}(0)\rangle-\langle\hat{V}(0)\hat{V}(\tau)\rangle\right)e^{i\Omega\tau}\right] \\
&= -\frac{1}{2\hbar}\left(\int_{-\infty}^{\infty} d\tau\langle\hat{V}(\tau)\hat{V}(0)\rangle e^{i\Omega\tau}-\int_{-\infty}^{\infty} d\tau\langle\hat{V}(0)\hat{V}(\tau)\rangle e^{i\Omega\tau}\right) \\
&= -\frac{1}{2\hbar}\left(S_{VV}(\Omega)-S_{VV}(-\Omega)\right).
\end{aligned} \tag{58}$$

According to Eq. (7), the impedance Z_p can be given by

$$Z_p = -\frac{\chi_v''(\Omega)}{\Omega} = \frac{1}{2\hbar\Omega}\left(S_{VV}(\Omega)-S_{VV}(-\Omega)\right), \tag{59}$$

which agrees with Eq. (47).

Here we assume the environment is in the thermal equilibrium and invoke the *detailed balance condition*:

$$S_{VV}(\Omega) = e^{\frac{\hbar\Omega}{k_\mathrm{B}T}}S_{VV}(-\Omega). \tag{60}$$

Plugging this into Eq. (47) or (59), the impedance becomes

$$Z_p = \frac{1}{2\hbar\Omega}\left(1-e^{-\frac{\hbar\Omega}{k_\mathrm{B}T}}\right)S_{VV}(\Omega). \tag{61}$$

In the classical limit ($T \gg \frac{\hbar\Omega}{k_\mathrm{B}}$) we have

$$Z_p \sim \frac{1}{2k_\mathrm{B}T}S_{VV}(\Omega), \tag{62}$$

and reproduce the classical result obtained in Eq. (15).

IV. PROBLEM

Let us consider the situation in which the system (LC circuit) and the environment (1D transmission line) are inductively coupled as opposed to the situation we have investigated, where the two are capacitively coupled (see Eq. (48)). The interaction Hamiltonian for the inductive coupling is given by

$$H_j = \hat{\Phi}_s(t)\hat{I}(t), \tag{63}$$

where $\hat{I}(t) = \hat{I}(x=0,t)$ is Schödinger's operator for the current at $x=0$, which can be wrtitten as

$$\begin{aligned} \hat{I}(t) &= \frac{1}{Z_p}\left(V^{\rightarrow}(0,t) - V^{\leftarrow}(0,t)\right) \\ &= \sigma_p\left(V^{\rightarrow}(0,t) - V^{\leftarrow}(0,t)\right), \end{aligned} \tag{64}$$

where

$$\sigma_p = \frac{1}{Z_p} \tag{65}$$

is the characteristic conductance of the 1D transmission line. Here $V^{\rightarrow}(x,t)$ and $V^{\leftarrow}(x,t)$ are defined in Eqs. (30) and (31), respectively. Let us here assume that $V^{\leftarrow}(0,t) = -V^{\rightarrow}(0,t)$ as for the closed terminal at $x=0$.

A. Current noise spectral density, $S_{II}(\Omega)$ [1, 2]

Prove that the current noise spectral density can be given by

$$\begin{aligned} S_{II}(\Omega) &= \int_{-\infty}^{\infty} d\tau \langle I(\tau)I(0)\rangle e^{i\Omega\tau} \\ &= \left(\frac{2\sigma_p\hbar\Omega}{1-e^{-\frac{\hbar\Omega}{k_{\mathrm{B}}T}}}\right). \end{aligned} \tag{66}$$

B. Kubo formula [1, 2]

Using the linear response theory with the interaction Hamiltonian H_j in Eq. (63), derive the expression for the conductance σ_p in terms of $S_{II}(\Omega)$ (*Kubo formula*):

$$\sigma_p = \frac{1}{2\hbar\Omega}\left(S_{II}(\Omega) - S_{II}(-\Omega)\right). \tag{67}$$

Confirm that plugging Eq. (66) into Eq. (67) produces an trivial equation.

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