Cavity QED and circuit QED

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We learned how two level systems can be treated quantum mechanically, how the electric dipole interaction emerged as the coupling term between a two level system and electromagnetic field, and deduce Einstein's A coefficient, a decay rate of the two level system due to the coupling to the *vacuum fluctuation* of electromagnetic field surrounding it. Here we learn how a cavity changes the *density of state* of the electromagnetic environment and modifies the decay rate of two level systems. We shall study the *Jaynes-Cummings model* in which an atom (or an artificial atom) is coupled to a cavity field. The coupled system exhibits a hybridized anharmonic energy-level structure called the *Jaynes-Cummings ladder*. The splitting of the nominally degenerate lowest two excited states is due to one and only one photon exchanged between two agencies, an atom and a cavity, and is called the *vacuum Rabi splitting*. We shall see how these interesting quantum phenomena modify under the realistic circumstances with ubiquitous dissipations.

I. PURCELL FACTOR

The number of modes within the bandwidth between $\omega + d\omega$ and ω for free space is:

$$N_A(\omega)d\omega = V\rho(\omega)\frac{d\omega}{2\pi} = \frac{V\omega^2}{\pi^2 c^3}d\omega, \tag{1}$$

while that for cavity

$$N_c(\omega)d\omega = \frac{\kappa}{2\pi}\frac{1}{((\omega-\omega_c)^2 + \left(\frac{\kappa}{2}\right)^2}, \tag{2}$$

where κ is the cavity decay rate and ω_c is the cavity resonance angular frequency. When resonance, we have

$$N_c(\omega)d\omega \xrightarrow{\omega\to\omega_c} \frac{2}{\pi}\frac{1}{\kappa} = \frac{2}{\pi}\frac{Q}{\omega_c}, \tag{3}$$

where the *quality factor* Q is defined by $Q = \frac{\omega_c}{\kappa}$.

The free-space spontaneous emission rate (Einsten A coefficient),

$$\Gamma_A(\omega) = 2\pi\,\frac{V\omega^2}{\pi^2 c^3}\,\frac{\mu^2}{3}\,\frac{\omega}{2\hbar\epsilon_0 V}, \tag{4}$$

can be modified if we place the cavity with quality factor of Q and volume of V as

$$\Gamma_c(\omega_c) = 2\pi\,\frac{2}{\pi}\frac{Q}{\omega_c}\,\mu^2\,\frac{\omega_c}{2\hbar\epsilon_0 V}, \tag{5}$$

We have thus the *Purcell factor*:

$$F_{\mathrm{P}} = \frac{\Gamma_c}{\Gamma_A} = \frac{3}{4\pi^2}\frac{Q}{V}\lambda^3, \tag{6}$$

which tells us how much the cavity enhances the spontaneous emission rate from the free-space Einsten A coefficient Γ_A. The Purcell factor becomes bigger by increasing Q and by decreasing the ratio $\frac{\lambda^3}{V}$.

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II. CAVITY QED

A. Jaynes-Cummings model

Let us consider the simplest possible scenario: An atom (a two level system) with energy $\hbar\omega_A$ and dipole moment,

$$\boldsymbol{d} = \frac{ea_0}{\sqrt{2}}\hat{\sigma}_x \boldsymbol{e}_x = \underbrace{ea_0}_{\mu}\left(\hat{J}^+ + \hat{J}^-\right)\boldsymbol{e}_x, \tag{7}$$

where $\hat{J}^+$ and $\hat{J}^-$ are

$$\hat{J}^+ = \frac{1}{\sqrt{2}}\left(\frac{\hat{\sigma}_x}{2} + i\frac{\hat{\sigma}_y}{2}\right) \tag{8}$$

$$\hat{J}^- = \frac{1}{\sqrt{2}}\left(\frac{\hat{\sigma}_x}{2} - i\frac{\hat{\sigma}_y}{2}\right), \tag{9}$$

interacts with a *Purcell-enhanced* electromagnetic mode in a cavity

$$\boldsymbol{E}_\perp = -i\sqrt{\frac{\hbar\omega_c}{2\epsilon_0 V}}\boldsymbol{e}_x\left(\hat{a} - \hat{a}^\dagger\right) \tag{10}$$

whose angular frequency ω_c is about the same as ω_A.

The celebrated *Jaynes-Cummings model* [1] is devised to depict this situation with an Hamiltonian

$$H_{JC} = \underbrace{\hbar\omega_A\hat{J}_3}_{\text{atom}} + \underbrace{\hbar\omega_c\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}_{\text{cavity field}} \underbrace{-i\hbar g\left(\hat{J}^+\hat{a} - \hat{J}^-\hat{a}^\dagger\right)}_{-\boldsymbol{d}\cdot\boldsymbol{E}_\perp}, \tag{11}$$

where $\hat{J}_3 = \frac{\hat{\sigma}_z}{2}$ and we took the *rotating-wave approximation.* Here the *vacuum Rabi angular frequency* can be defined by

$$g \equiv \frac{\mu\sqrt{\frac{\hbar\omega_c}{2\epsilon_0 V}}}{\hbar} = \mu\sqrt{\frac{\omega_c}{2\epsilon_0\hbar V}} \equiv \frac{\Omega_0}{2}. \tag{12}$$

Note that the dipole moment μ in Eq. (12) and the dipole moment of the circular Rydberg atom μ_1 are related by $\mu = \sqrt{2}\mu_1$. We can notice that the electric dipole interaction term in Eq. (11) only connects $|0\rangle_A|n\rangle_c$ state and $|1\rangle_A|n-1\rangle_c$ state and that H_{JC} can be block-diagonalize like

$$H_{JC} = \begin{bmatrix} \ddots & & & & & \\ & * & * & & & \\ & * & * & & & \\ & & & * & * & \\ & & & * & * & \\ & & & & & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & & \\ & H_n & & \\ & & H_{n+1} & \\ & & & \ddots \end{bmatrix}. \tag{13}$$

Within one manifold, we have

$$H_n = \hbar\begin{bmatrix} \frac{1}{2}\omega_A + \left(n - \frac{1}{2}\right)\omega_c & -ig\sqrt{n} \\ ig\sqrt{n} & -\frac{1}{2}\omega_A + \left(n + \frac{1}{2}\right)\omega_c \end{bmatrix}. \tag{14}$$

For the resonant case in which $\omega_A = \omega_c \equiv \omega_0$, the Hamiltonian Eq. (14) becomes

$$H_n = \hbar\begin{bmatrix} n\omega_0 & -ig\sqrt{n} \\ ig\sqrt{n} & n\omega_0 \end{bmatrix}. \tag{15}$$

The eigenenergies are obtained to be

$$E_n^\pm = \hbar\omega_0 n \pm \hbar g\sqrt{n}, \tag{16}$$

with respective eigenstates

$$|\psi_{\pm}^{(n)}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|n\rangle_c \ \mp\ i|1\rangle_A|n-1\rangle_c\right). \tag{17}$$

The *anharmonic* energy level structure for atom-cavity system given by Eq, (16) is called the *Jaynes-Cummings ladder*, where the nominally degenerate energy eigenstates $|0\rangle_A|n\rangle_c$ and $|1\rangle_A|n-1\rangle_c$ are hybridized to become non-degenerate energy eigenstates $|\psi_{\pm}^{(n)}\rangle$ whose energies are split by $2\hbar g\sqrt{n} = \Omega_0\sqrt{n}$. In particular the splitting occurs even when $n = 1$, that is, the relevant manifold spanned by $|0\rangle_A|1\rangle_c$ and $|1\rangle_A|0\rangle_c$. The energy splitting $2\hbar g = \hbar\Omega_0$ is called the *vacuum Rabi splitting*, with which the atom and the cavity field exchange one and only one photon at the rate of Ω_0.

For the general case in which $\omega_A - \omega_c \equiv \Delta$, the Hamiltonian Eq. (14) becomes

$$H_n = \hbar \begin{bmatrix} n\omega_c - \frac{\Delta}{2} & -ig\sqrt{n} \\ ig\sqrt{n} & n\omega_c + \frac{\Delta}{2} \end{bmatrix}, \tag{18}$$

and the eigenenergies are

$$E_n^{\pm} = \hbar\omega_c n \pm \hbar\sqrt{\frac{\Delta^2}{4} + g^2 n}, \tag{19}$$

with eigenstates

$$|\psi_{\pm}^{(n)}\rangle = \frac{1}{\sqrt{A_{\pm}}}\left(\left(\frac{\Delta}{2} \pm \sqrt{\frac{\Delta^2}{4} + g^2 n}\right)|0\rangle_A|n\rangle_c \ - \ ig\sqrt{n}\,|1\rangle_A|n-1\rangle_c\right), \tag{20}$$

where

$$A_{\pm} = \frac{\Delta^2}{2} + 2g^2 n \pm \Delta\sqrt{\frac{\Delta^2}{4} + g^2 n}. \tag{21}$$

B. Bloch-Heisenberg-Langevin equations

To be more realistic we shall now incorporate the dissipations both for the atom and the cavity field and see the conditions in which the spectacular quantum phenomena, such as the vacuum Rabi splitting, predicted by the Jaynes-Cummings model can be observed. By adding the extra terms responsible for the dissipations to the Jaynes-Cummings Hamiltonian Eq. (11) we have

$$\begin{aligned} H \ &= \ \underbrace{\hbar\omega_A \hat{J}_3 + \hbar\omega_c\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) - i\hbar g\left(\hat{J}^+\hat{a} - \hat{J}^-\hat{a}^\dagger\right)}_{H_{JC}}, \\ &- \ i\hbar\sqrt{\Gamma_A}\int\frac{d\omega}{2\pi}\left(\hat{J}^+\hat{b}(\omega) - \hat{J}^-\hat{b}^\dagger(\omega)\right) + \int\frac{d\omega}{2\pi}\hbar\omega\hat{b}^\dagger(\omega)\hat{b}(\omega) \\ &- \ i\hbar\sqrt{\kappa}\int\frac{d\omega}{2\pi}\left(\hat{a}^\dagger\hat{c}(\omega) - \hat{a}\hat{c}^\dagger(\omega)\right) + \int\frac{d\omega}{2\pi}\hbar\omega\hat{c}^\dagger(\omega)\hat{c}(\omega), \end{aligned} \tag{22}$$

where Γ_A represents the atomic decay rate (the Einstein A coefficient) to the free-space continuum field $\hat{b}(\omega)$ and κ represents the cavity field decay rate to the continuum field $\hat{c}(\omega)$.

Now let us suppose that the atomic state rests predominantly on $|0\rangle\langle 0|$ as is the case where all the relevant field amplitudes are very small and thus

$$\langle 2\hat{J}_3\rangle = -1. \tag{23}$$

This is in fact the trick that make the otherwise nonlinear coupled equations linear ones in the end. What we have to do is then to find the equations of motion for the cavity field $\hat{a}$ and the atomic dipole $\hat{J}^-$. For $\hat{a}$ we have

$$\dot{\hat{a}} = \frac{i}{\hbar}[H, \hat{a}] = -i\omega_c\hat{a} + g\hat{J}^- - \sqrt{\kappa}\int\frac{d\omega}{2\pi}\hat{c}(\omega), \tag{24}$$

and for $\hat{J}^-$ we have

$$\dot{\hat{J}}^- = \frac{i}{\hbar}\left[H, \hat{J}^-\right] = -i\omega_A\hat{a} + g\underbrace{2\hat{J}_3}_{-1}\hat{a} - \sqrt{\Gamma_A}\int\frac{d\omega}{2\pi}\hat{b}(\omega)$$
$$= -i\omega_A\hat{a} - g\hat{a} - \sqrt{\Gamma_A}\int\frac{d\omega}{2\pi}\hat{b}(\omega). \quad (25)$$

The bath modes $\hat{b}(\omega)$ and $\hat{c}(\omega)$ can be formally solved to be

$$\hat{b}(\omega, t) = e^{-i\omega(t-t_0)}\hat{b}(\omega, t_0) + \sqrt{\Gamma_A}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{J}^-(\tau) \quad (26)$$
$$\hat{c}(\omega, t) = e^{-i\omega(t-t_0)}\hat{c}(\omega, t_0) + \sqrt{\kappa}\int_{t_0}^{t} d\tau e^{-i\omega(t-\tau)}\hat{a}(\tau). \quad (27)$$

These are plugging into Eqs. (24) and (25) to have the coupled equations of motion in a rotating frame at a frequency ω_0:

$$\dot{\hat{a}} = -i\left(\underbrace{\omega_c + \Delta}_{\omega_c'} - \omega_0\right)\hat{a} + g\hat{J}^- - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{c}(t) \quad (28)$$
$$\dot{\hat{J}}^- = -i\left(\underbrace{\omega_A + \Delta + \Delta'}_{\omega_A'} - \omega_0\right)\hat{J}^- - g\hat{a} - \frac{\Gamma_A}{2}\hat{J}^- - \sqrt{\Gamma_A}\hat{b}(t), \quad (29)$$

where $\hat{c}(t)$ and $\hat{b}(t)$ are the time-domain continuum (input) field operators:

$$\hat{c}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{c}(\omega)e^{-i(\omega-\omega_0)t} \quad (30)$$
$$\hat{b}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{b}(\omega)e^{-i(\omega-\omega_0)t}. \quad (31)$$

The equations (28) and (29) can be called the *Bloch-Heisenberg-Langevin equations*, which generalize the Jaynes-Cummings model for including the dissipations and the resultant fluctuations, that is, the Langevin forces appeared in the last terms in Eqs. (28) and (29).

C. Weak coupling regime and strong coupling regime [3]

Neglecting the input noise fields $\hat{c}(t)$ and $\hat{b}(t)$, we have the following coupled equations:

$$\begin{bmatrix}\dot{\hat{a}} \\ \dot{\hat{J}}^-\end{bmatrix} = \begin{bmatrix} -\frac{\kappa}{2}(1+i\Theta) & g \\ -g & -\frac{\Gamma_A}{2}(1+i\Delta)\end{bmatrix}\begin{bmatrix}\hat{a} \\ \hat{J}^-\end{bmatrix}, \quad (32)$$

where

$$\Theta = \frac{\omega_c' - \omega_0}{\kappa/2} \quad (33)$$
$$\Delta = \frac{\omega_A' - \omega_0}{\Gamma_A/2} \quad (34)$$

are the cavity detuning parameter and the atomic detuning parameter, respectively.

To gain some insights without mess let us consider the case where $\Theta = \Delta = 0$. The matrix in Eq. (32) can be diagonalized for the respective hybridized eigenoperators with the eigenvalues

$$\lambda_\pm = -\frac{\kappa + \Gamma_A}{4} \pm \sqrt{\left(\frac{\kappa - \Gamma_A}{4}\right)^2 - g^2}, \quad (35)$$

where the real parts of the eigenvalues $\lambda_\pm$ represent the decays while the imaginary parts represent the detunings from ω_0. We shall now examine some limiting cases.

1. *Weak coupling regime*

First, consider the weak coupling regime $g \ll \kappa, \Gamma_A$. There are then two symmetric limits. The one is the broad cavity limit (bad cavity limit), $\kappa \gg \Gamma_A$, in which the eigenvalues becomes

$$\lambda_+ \simeq -\frac{\Gamma_A}{2}\left(1+\frac{4g^2}{\kappa\Gamma_A}\right) = -\frac{\Gamma_A}{2}\left(1+\mathcal{C}\right) \tag{36}$$

$$\lambda_- \simeq -\frac{\kappa}{2}\left(1-\frac{4g^2}{\kappa\Gamma_A}\right) = -\frac{\kappa}{2}\left(1-\mathcal{C}\right), \tag{37}$$

where $\mathcal{C} = \frac{4g^2}{\kappa\Gamma_A}$ is called *cooperativity* and is omnipresent whenever the two dissipative systems couple. Note that the cooperativity coincides with the Purcell factor:

$$\mathcal{C} = \frac{4g^2}{\kappa\Gamma_A} = \frac{4\left(\sqrt{\mu^2\frac{\omega}{2\hbar\epsilon_0 V}}\right)^2}{\left(\frac{\omega}{Q}\right)\left(2\pi\frac{V\omega^2}{\pi^2c^3}\frac{\mu^2}{3}\frac{\omega}{2\hbar\epsilon_0 V}\right)} = \frac{3}{4\pi^2}\frac{Q}{V}\lambda^3 = F_{\mathrm{p}}. \tag{38}$$

In the weak coupling regime λ_+ is more or less the eigenvalue of the atomic operator $\hat{J}^-$ and expresses the well-known *Purcell-enhancement* of the spontaneous emission due to the presence of the cavity, while λ_- is more or less the eigenvalue of the cavity field operator $\hat{a}$ and expresses the lesser known *atom-inhibited* cavity decay. The other limiting case within the weak coupling regime is the broad atomic response limit, $\Gamma_A \gg \kappa$, with the eigenvalues

$$\lambda_+ \simeq -\frac{\kappa}{2}\left(1+\mathcal{C}\right) \tag{39}$$

$$\lambda_- \simeq -\frac{\Gamma_A}{2}\left(1-\mathcal{C}\right). \tag{40}$$

Now λ_+ is more or less the eigenvalue of the cavity field operator $\hat{a}$ and expresses the *atom-enhanced* cavity decay, while λ_- is more or less the eigenvalue of the atomic operator $\hat{J}^-$ and expresses the *cavity-inhibited* spontaneous emission.

2. *Strong coupling regime*

When the coupling strength g becomes greater than the cavity decay rate κ and the atomic decay rate Γ_A, that is, $g > \kappa, \Gamma_A$, the eigenvalues in Eq. (35) becomes complex values

$$\lambda_\pm = -\frac{\kappa+\Gamma_A}{4} \pm i\Omega_0, \tag{41}$$

where

$$\Omega_0 = \sqrt{g^2 - \left(\frac{\kappa-\Gamma_A}{4}\right)^2}. \tag{42}$$

Figures 1 and 2 respectively depict the real and imaginary part of $\lambda_\pm$ in Eq. (41) as functions of the coupling strength g. When the coupling strength g reaches the value $|\kappa-\Gamma_A|/4$ the nominally different decay rates of the eigenoperators are merged and become a unique one,

$$\mathrm{Im}[\lambda_\pm] = \frac{|\kappa-\Gamma_A|}{4}, \tag{43}$$

while the nominally degenerate (zero) detunings bifurcate causing the *vacuum Rabi splitting*, a hallmark of the strong coupling regime where the spectacular quantum phenomena predicted by the Jaynes-Cummings model can be seen even with the dissipations! In other words, the dissipations suppress the vacuum Rabi splitting when the dissipations overwhelme the coherent coupling.

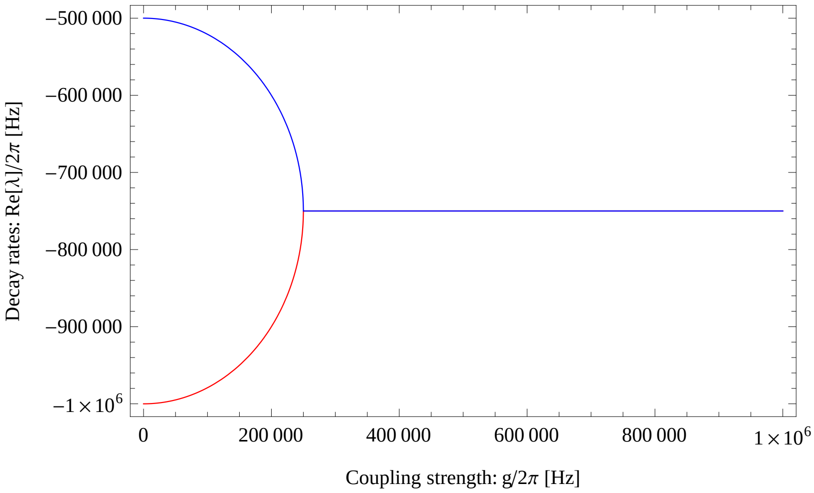

FIG. 1: The real part of the complex eigenvalues $\lambda_\pm$ (the decay rates of the eigenoperators consisting of $\hat{a}$ and $\hat{J}^-$) as a function of the coupling strength g. Here $\kappa/2\pi = 2$ MHz and $\Gamma_A/2\pi = 1$ MHz. We can see at $g = 250$ kHz the decay rates merged entering the strong coupling regime.

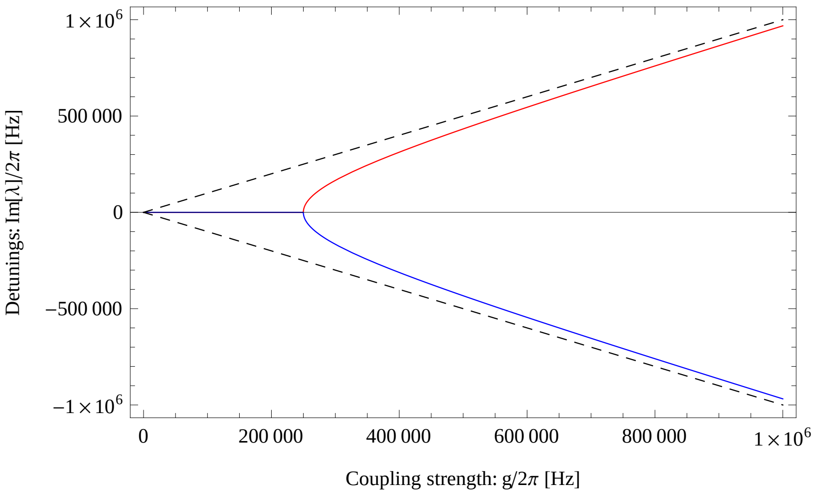

FIG. 2: The imaginary part of the complex eigenvalues $\lambda_\pm$ (the detunings of the eigenoperators consisting of $\hat{a}$ and $\hat{J}^-$) as a function of the coupling strength g. Here $\kappa/2\pi = 2$ MHz and $\Gamma_A/2\pi = 1$ MHz. We can see at $g = 250$ kHz the nominally degenerate detunings bifurcate and further increase of g causes the vacuum Rabi splitting, a hallmark of the strong coupling regime. Note that the dashed lines indicate the vacuum Rabi splitting in the case of no dissipations.

III. CIRCUIT QED [4]

It is apparent that the coupling strength g can be boosted by shrinking the cavity volume V. If we push the volume V towards the ultimately small one $V \sim \lambda\langle x\rangle^2$ for the dipole moment of $\mu = e\langle x\rangle$ with λ being the wavelength of the relevant field, we have the following limit on the coupling strength:

$$g_c = e\langle x\rangle\sqrt{\frac{\omega}{2\hbar\epsilon_0\lambda\langle x\rangle^2}} = \sqrt{\frac{e^2}{4\pi\epsilon_0\hbar c}}\omega = \sqrt{\alpha}\omega, \tag{44}$$

where α is the fine-structure constant. For the natural atoms $\langle x\rangle$ is roughly the Bohr radius (~ 0.1 nm) and for the relevant optical transition λ is roughly 1μm and thus building such an ultimately small cavity is very hard, if not impossible. For the artificial atoms with Josephson junctions, on the other hand, $\langle x\rangle$ can be designed to be very large (typically $\sim 100\mu$m). The relevant transition $\omega/2\pi$ for the typical Josephson atom is around 10GHz (which should be far smaller than the superconducting gap energy) and λ is accordingly roughly 3 cm. These two facts puts the current micro-fabrication technology in a favorable position to build a (near-field) 1D transmission-line cavity which achieves the ultimate volume $V \sim \lambda \langle x \rangle^2$. We have been witnessing the spectacular developments of such *circuit QED* systems with an artificial Josephson atom and a microwave cavity.

IV. PROBLEM

A. Artificial atom with Josephson junctions [5]

Reading Ref. [5] and show that the Josephson junction atom (nonlinear LC circuit where L is replaced by a Josephson junction) can be considered to have an electric dipole moment of

$$\mu_J = \left(\frac{2E_J}{E_{CP}} \right)^{\frac{1}{4}} e, \tag{45}$$

where E_J is the Josephson energy and E_{CP} is the charging energy given by $E_{CP} = \frac{(2e)^2}{C_J}$ with C_J being the capacitance of the Josephson junction. By finding the typical values of E_J and E_{CP} in the literatures and evaluate the μ_J. Compare it to the value for classical electron oscillator,

$$\mu_1 = e x_{\mathrm{zpf}} = \frac{e a_0}{\sqrt{2}}, \tag{46}$$

introduced in Eq. (102) in *note 2015-12-21* and calculate the oscillator strength of the Josephson junction atom,

$$\mathcal{F} = \frac{1}{3} \left(\frac{\mu_J}{\mu_1} \right)^2. \tag{47}$$

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