Harmonic oscillators, coupled harmonic oscillators, and Bosonic fields

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We first study why harmonic oscillators are so ubiquitous and see that not only a point mass in a harmonic potential but also an LC circuit behave like a harmonic oscillator. We then learn an important idea of *normal modes* to deal with coupled harmonic oscillators. Finally we see that taking the *continuum limit* a (1+1)-dimensional Bosonic field is emerged from coupled (0+1)-dimensional harmonic oscillators. As examples we deal with a longitudinal acoustic phonon mode in a 1D atomic chain and an itinerant microwave photon mode propagating in a 1D transmission line.

I. HARMONIC OSCILLATORS

A. Point mass in a harmonic potential

1. *Lagrangian and Hamiltonian formalism [1, 2]*

Let us begin by considering a point mass with mass m and coordinate x situated in a potential $U(x)$. Suppose that the mass is oscillating with small amplitude around the equilibrium position x_0. Then the potential energy of the mass can be Taylor-expanded around x_0:

$$U(x) = U(x_0) + \frac{\partial U(x_0)}{\partial x} x + \frac{1}{2}\frac{\partial^2 U(x_0)}{\partial x^2} x^2. \tag{1}$$

Since the force, $F = \frac{\partial U(x_0)}{\partial x}$, should be zero in the equilibrium position, neglecting the potential offset $U(x_0)$ we have

$$U(x) = \frac{1}{2}\underbrace{\frac{\partial^2 U(x_0)}{\partial x^2}}_{k} x^2. \tag{2}$$

This suggests that any potential can be considered as a harmonic potential when we are interested in the small motion in the vicinity of the equilibrium position.

With the kinetic part $K = \frac{1}{2}m\dot{x}^2$ we have the standard Lagrangian for the harmonic oscillator:

$$L(x,\dot{x}) = K(\dot{x}) - U(x) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{3}$$

The motion of the mass from time t_1 to t_2 can be determined so as to minimize the action integral

$$I = \int_{t_1}^{t_2} L(x,\dot{x})dt. \tag{4}$$

The minimum of I can be obtained by a *variational principle (Hamilton's principle)*, which leads to the Euler-Lagrange equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L(x,\dot{x})}{\partial \dot{x}}\right) - \frac{\partial L(x,\dot{x})}{\partial x} = 0, \tag{5}$$

that nothing but Newton's second law:

$$m\ddot{x} - kx = 0. \tag{6}$$

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In the formal procedure the conjugate momentum, p, can then be obtained by

$$p = \frac{\partial L(x, \dot{x})}{\partial \dot{x}} = m\dot{x}. \tag{7}$$

We have thus the Hamiltonian $H(x, p)$ from the Legendre transformation:

$$H(x, p) = \dot{x}p - L(x, \dot{x}) = \frac{1}{2m}p^2 + \frac{1}{2}kx^2 = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{8}$$

where $\omega = \sqrt{\frac{k}{m}}$ will trun out to be the eigen angular frequency of the oscillator.

2. *Canonical quantization*

We can promote x and p to the quantum-mechanical operators by imposing the commutation relation,

$$[\hat{x}, \hat{p}] = i\hbar. \tag{9}$$

Let the annihilation and creation operators be

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega}\hat{p} \right) \tag{10}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega}\hat{p} \right), \tag{11}$$

respectively. The Hamiltonian Eq. (8) can then be written in a diagonal form as

$$\begin{aligned} \hat{H}(\hat{x}, \hat{p}) &= \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 \\ &= \hbar\omega \left(\underbrace{\hat{a}^\dagger \hat{a}}_{\hat{n}} + \frac{1}{2} \right). \end{aligned} \tag{12}$$

The mean value of energy in thermal equilibrium $\langle \hat{H}(\hat{x}, \hat{p}) \rangle = \mathrm{Tr}[\rho \hat{H}]$ becomes

$$\langle \hat{H}(\hat{x}, \hat{p}) \rangle = \hbar\omega \left(\langle \hat{n} \rangle + \frac{1}{2} \right), \tag{13}$$

where

$$\langle n \rangle = \frac{1}{e^{\frac{\hbar\omega}{k_{\mathrm{B}}T}} - 1}, \tag{14}$$

which in high temperature limit becomes $\langle n \rangle \to \frac{k_{\mathrm{B}}T}{\hbar\omega}$.

B. LC circuit [3]

1. *Q-representation (loop variable representation)*

Let us see that an LC circuit can also be viewed as a harmonic oscillator. Replacing the coordinate x by the charge Q, the mass m by the inductance L_0, and the spring constant k by the inverse of capacitance C_0, we have the Lagrangian for the LC circuit:

$$L(Q, \dot{Q}) = \frac{1}{2}L_0\dot{Q}^2 - \frac{1}{2C_0}Q^2. \tag{15}$$

The first term is the inductive energy and the second is the charging energy. The Euler-Lagrange equation of motion can then be given by

$$L_0\ddot{Q} - \frac{Q}{C_0} = 0, \tag{16}$$

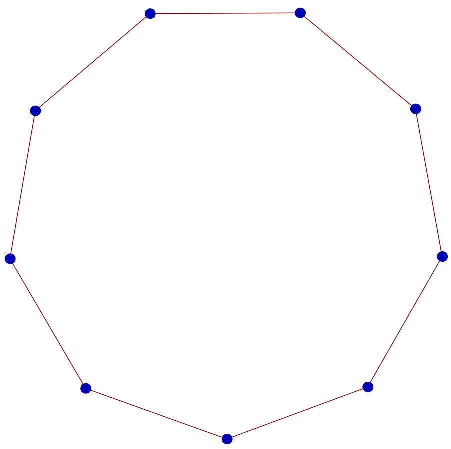

FIG. 1: The *first Kirchhoff's law* states that the sum of the voltages around a *loop* of a circuit such as shown here is equal to zero. A series LC circuit connected to a resistor (see Fig. 3 (a)) has such a loop.

which shows the sum of the voltages ($\sim$ *velocities*) around a *loop* of a circuit is equal to zero (*first Kirchhoff's law*, see Fig. 1).

The conjugate momentum is

$$\frac{\partial L(Q,\dot{Q})}{\partial \dot{Q}} = L_0\dot{Q} = \varphi, \tag{17}$$

which is identified as the flux. The Hamiltonian is thus

$$\begin{aligned} H(Q,\varphi) = \dot{Q}\varphi - L(Q,\dot{Q}) &= \frac{1}{2L_0}\varphi^2 + \frac{1}{2C_0}Q^2 \\ &= \frac{1}{2L_0}\varphi^2 + \frac{1}{2}L_0\omega^2 Q^2, \end{aligned} \tag{18}$$

where $\omega = \frac{1}{\sqrt{L_0C_0}}$.

The commutation relation:

$$[Q,\varphi] = i\hbar \tag{19}$$

Let the annihilation and creation operators be

$$\hat{b} = \sqrt{\frac{L_0\omega}{2\hbar}}\left(\hat{Q} + \frac{i}{L_0\omega}\hat{\varphi}\right) \tag{20}$$

$$\hat{b}^\dagger = \sqrt{\frac{L_0\omega}{2\hbar}}\left(\hat{Q} - \frac{i}{L_0\omega}\hat{\varphi}\right). \tag{21}$$

The Hamiltonian Eq. (18) can then be written as

$$\begin{aligned} \hat{H}(\hat{Q},\hat{\varphi}) &= \frac{1}{2L_0}\hat{\varphi}^2 + \frac{1}{2}L_0\omega^2\hat{Q}^2 \\ &= \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right). \end{aligned} \tag{22}$$

2. φ-representation (node variable representation)

We can equally use the flux φ as the coordinate of the LC circuit system, which is more relevant when we deal with a transmission line. Then, the Lagrangian is a function of φ and $\dot{\varphi}$, and is given by

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} C_0 \dot{\varphi}^2 - \frac{1}{2L_0} \varphi^2, \tag{23}$$

where the roles of C_0 and L_0^{-1} are the mass and the spring constant, respectively, and are switched from the first case. The first term is then the charging energy and the second is the inductive energy. The Euler-Lagrange equation of motion can then be given by

$$C_0 \ddot{\varphi} - \frac{\varphi}{L_0} = 0, \tag{24}$$

which shows the sum of the currents ($\sim$ *forces*) arriving at a *node* of a circuit is equal to zero (*second Kirchhoff's law*, see Fig. 2).

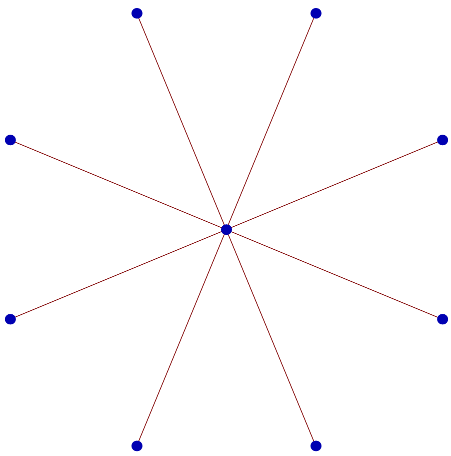

FIG. 2: The *second Kirchhoff's law* states that the sum of the currents arriving at a *node* of a circuit such as shown here is equal to zero. A parallel LC circuit connected to a resistor (see Fig. 3 (b)) has such a node.

The conjugate momentum becomes

$$\frac{\partial L(\varphi, \dot{\varphi})}{\partial \dot{\varphi}} = C_0 \dot{\varphi}. \tag{25}$$

Since $\dot{\varphi} = L_0 \dot{I} = V$ (*Faraday's law of induction*) the conjugate momentum of the flux φ is indeed the charge:

$$C_0 \dot{\varphi} = C_0 V = Q. \tag{26}$$

Consequently, the Hamiltonian is

$$\begin{aligned} H(\varphi, Q) = \dot{\varphi} Q - L(\varphi, \dot{\varphi}) &= \frac{1}{2C_0} Q^2 + \frac{1}{2L_0} \varphi^2 \\ &= \frac{1}{2C_0} Q^2 + \frac{1}{2} C_0 \omega^2 \varphi^2. \end{aligned} \tag{27}$$

The commutation relation:

$$[\varphi, Q] = i\hbar \tag{28}$$

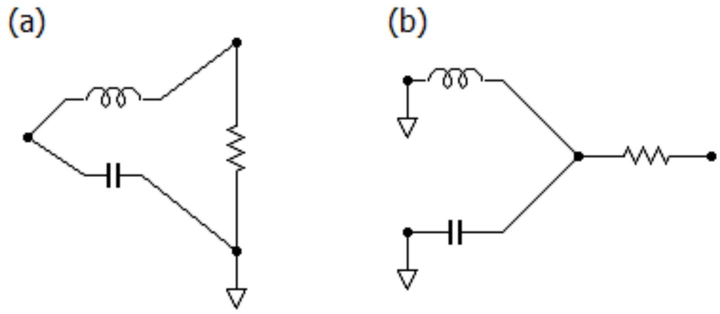

FIG. 3: (a) A series LC circuit connected to a resistor. (b) A parallel LC circuit connected to a resistor.

TABLE I: Point mass - LC circuit - EM cavity mode

	Point mass	LC (Q-rep./ loop rep.)	LC (φ-rep./ node rep.)	EM cavity mode
Mass	m	L_0	C_0	ϵ_0
Spring const.	k	$\frac{1}{C_0}$	$\frac{1}{L_0}$	$\frac{1}{\mu_0}$
Ang. freq.	$\omega = \sqrt{\frac{k}{m}}$	$\omega = \frac{1}{\sqrt{C_0 L_0}}$	$\omega = \frac{1}{\sqrt{C_0 L_0}}$	$\omega_k = ck$
Position	x	Q	φ	A_k
Velocity	$\dot{x} = v$	$\dot{Q} = I$	$\dot{\varphi} = V$	$\dot{A}_k = -E_k$
Momentum	$p = m\dot{x} = mv$	$\varphi = L_0\dot{Q} = L_0 I$	$Q = C_0\dot{\varphi} = C_0 V$	$\Pi_k = \epsilon_0 \dot{A}_k = -\epsilon_0 E_k$
Force	$F = \dot{p} = kx$	$V = \dot{\varphi} = \frac{Q}{C_0}$	$I = \dot{Q} = \frac{\varphi}{L_0}$	$\dot{\Pi}_k = \frac{A_k}{\mu_0}$

Let the annihilation and creation operators be

$$\hat{c} = \sqrt{\frac{C_0\omega}{2\hbar}}\left(\hat{\varphi} + \frac{i}{C_0\omega}\hat{Q}\right) \tag{29}$$

$$\hat{c}^\dagger = \sqrt{\frac{C_0\omega}{2\hbar}}\left(\hat{\varphi} - \frac{i}{C_0\omega}\hat{Q}\right). \tag{30}$$

The Hamiltonian Eq. (27) can then be written as

$$\begin{aligned} \hat{H}(\hat{\varphi}, \hat{Q}) &= \frac{1}{2C_0}\hat{Q}^2 + \frac{1}{2}C_0\omega^2\hat{\varphi}^2 \\ &= \hbar\omega\left(\hat{c}^\dagger\hat{c} + \frac{1}{2}\right). \end{aligned} \tag{31}$$

Compared with the point mass case, the Q-representation chooses Q as a position variable while φ-representation chooses φ for that. In view of standard circuit theory, these choices are related to Kirchhoff's laws; the former chooses loop currents as a set of independent degrees of freedom of the circuit while the latter chooses node voltages for that [3]. We will mainly use the latter φ-representation, which has more direct relevance to 1D transmission line when taking continuum limit of coupled LC circuits.

Table I shows the correspondances of the physical quantities of point mass, LC circuit, as well as electromagnetic cavity mode, all of which can be viewed as a harmonic oscillator.

II. COUPLED HARMONIC OSCILLATORS

What happens if we couple two harmonic oscillators? Here, the coupling means that the two oscillators exchange energy. The coupling can be incorporated in the Hamiltonian as a quadratic term, which contains dynamical variables (i.e., coordinates and momenta) of both oscillators. To see more detail, let us take a look at a system where a mechanical oscillator is coupled with an LC circuit, an *electro-mechanical* system.

A. Electro-mechanics

Let us consider the situation in which a metallic membrane oscillator with the angular frequency of ω_m is capacitively coupled to a LC circuit with the angular frequency of ω_{LC}. The coupled system's potential can then be given by $H(x, Q)$, where x is the membrane displacement and Q is the charge in the capacitor of the LC circuit. Suppose that with certain external voltage the equilibrium position is $x = X_0$, and the equilibrium charge is $Q = Q_0$. Then, around the equilibrium point, the potential can be written as

$$H(x,Q) = H(X_0,Q_0) + (\frac{\partial H}{\partial x}\hat{x} + \frac{\partial H}{\partial Q}\hat{Q}) + (\frac{1}{2}\frac{\partial^2 H}{\partial x^2}\hat{x}^2 + \frac{1}{2}\frac{\partial^2 H}{\partial Q^2}\hat{Q}^2 + \frac{\partial^2 H}{\partial x \partial Q}\hat{x}\hat{Q}), \tag{32}$$

with $x = X_0+\hat{x}, Q = Q_0+\hat{Q}$. The linear terms in Eq. (32), however, vanish becasue of the definition of the equilibrium condition $\frac{\partial H}{\partial x}|_{x=X_0} = 0$ and $\frac{\partial H}{\partial Q}|_{Q=Q_0} = 0$. Neglecting the equilibrium potential energy $H(X_0, Q_0)$, we have

$$\hat{H}(\hat{x},\hat{Q}) = \frac{1}{2}m\omega_m\hat{x}^2 + \frac{1}{2C}\hat{Q}^2 + G\hat{x}\hat{Q}, \tag{33}$$

where $\frac{\partial^2 H}{\partial x^2}|_{x=X_0} = k = m\omega_m^2$, $\frac{\partial^2 H}{\partial Q^2}|_{Q=Q_0} = \frac{1}{C}$, and $\frac{\partial^2 H}{\partial x \partial Q}|_{x=X_0,Q=Q_0} = G$.

By adding the kinetic energy parts, $\frac{1}{2m}p^2 = \frac{1}{2}m(\frac{dx}{dt})^2$ for mechanics and $\frac{1}{2L}\phi^2 = \frac{1}{2}L(\frac{dQ}{dt})^2$ for LC-circuit, we have the Hamiltonian,

$$\hat{H} = \underbrace{\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_m\hat{x}^2}_{\text{mechanics}} + \underbrace{\frac{1}{2L}\hat{\phi}^2 + \frac{1}{2C}\hat{Q}^2}_{\text{LC}} + \underbrace{G\hat{x}\hat{Q}}_{\text{coupling}}\ . \tag{34}$$

The Hamiltonian Eq. (34) can be rewritten in terms of the annihilation and the creation operators as

$$\begin{aligned}
\hat{H} &= \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_m\hat{x}^2 + \frac{1}{2L}\hat{\phi}^2 + \frac{1}{2}L\omega_{LC}\hat{Q}^2 + G\hat{x}\hat{Q} \\
&= \hbar\omega_m\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_{LC}\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + G\left(\sqrt{\frac{\hbar}{2m\omega_m}}\left(\hat{a}^\dagger + \hat{a}\right)\right)\left(\sqrt{\frac{\hbar}{2L\omega_{LC}}}\left(\hat{b}^\dagger + \hat{b}\right)\right) \\
&= \hbar\omega_m\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_{LC}\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \frac{\hbar}{2}\underbrace{\frac{G}{\sqrt{m\omega_m}\sqrt{L\omega_{LC}}}}_{g}\left(\hat{a}^\dagger + \hat{a}\right)\left(\hat{b}^\dagger + \hat{b}\right).
\end{aligned} \tag{35}$$

Using the *rotating-wave approximation* which neglect rapidly oscillating terms $\hat{a}\hat{b}$ and $\hat{a}^\dagger\hat{b}^\dagger$ in the last term in Eq. (35) we have the *canonical* Hamiltonian for the coupled oscillator system:

$$\hat{H} = \hbar\omega_m\hat{a}^\dagger\hat{a} + \hbar\omega_{LC}\hat{b}^\dagger\hat{b} + \frac{\hbar}{2}g\left(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\right). \tag{36}$$

Here the vacuum energy terms are omitted since the energy can be offset arbitrary.

Let us analyze the energy level structure for the coupled system. First, suppose that the two oscillators are resonant, that is, $\omega_m = \omega_{LC} = \omega$. We then easily guess that the *normal modes*, which diagonalize the Hamiltonian Eq. (36), are

$$\hat{c} = \frac{1}{\sqrt{2}}\left(\hat{a} - \hat{b}\right) \tag{37}$$

$$\hat{d} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{b}\right). \tag{38}$$

With these normal mode operators the Hamiltonian can be indeed rewritten in a diagonal form as

$$\hat{H} = \left(\hbar\omega - \frac{\hbar g}{2}\right)\hat{c}^{\dagger}\hat{c} + \left(\hbar\omega + \frac{\hbar g}{2}\right)\hat{d}^{\dagger}\hat{d}. \tag{39}$$

The eigen-energies are shifted from the originally degenerate $\hbar\omega$ by $\hbar g$, which called *normal mode splitting*.

Next, let us consider the situation where the mechanical and LC oscillators have different resonance angular frequencies, ω_m and $\omega_{LC} = \omega_m + \Delta$, respectively. The normal modes in this case become

$$\hat{c} \;=\; \cos\theta\;\hat{a} - \sin\theta\;\hat{b} \tag{40}$$

$$\hat{d} \;=\; \sin\theta\;\hat{a} + \cos\theta\;\hat{b}, \tag{41}$$

where the *mixing angle* θ is defined by

$$\cot 2\theta = \frac{\Delta}{\omega}. \tag{42}$$

The resultant diagonalized Hamiltonian is

$$\begin{aligned}\hat{H} \;&=\; \left(\underbrace{\hbar\omega_m + \frac{\hbar\Delta}{2}}_{\frac{1}{2}(\omega_m+\omega_{LC})} - \frac{\hbar g}{2}\frac{1}{\sin 2\theta}\right)\hat{c}^{\dagger}\hat{c} + \left(\underbrace{\hbar\omega_m + \frac{\hbar\Delta}{2}}_{\frac{1}{2}(\omega_m+\omega_{LC})} + \frac{\hbar g}{2}\frac{1}{\sin 2\theta}\right)\hat{d}^{\dagger}\hat{d} \\ &=\; E_1(\Delta)\hat{c}^{\dagger}\hat{c} + E_2(\Delta)\hat{d}^{\dagger}\hat{d}.\end{aligned} \tag{43}$$

The eigen-energies E_1 and E_2 for the nornal modes $\hat{c}$ and $\hat{d}$ are thus varies as a function of Δ. Figure 4 shows such a normal mode splitting.

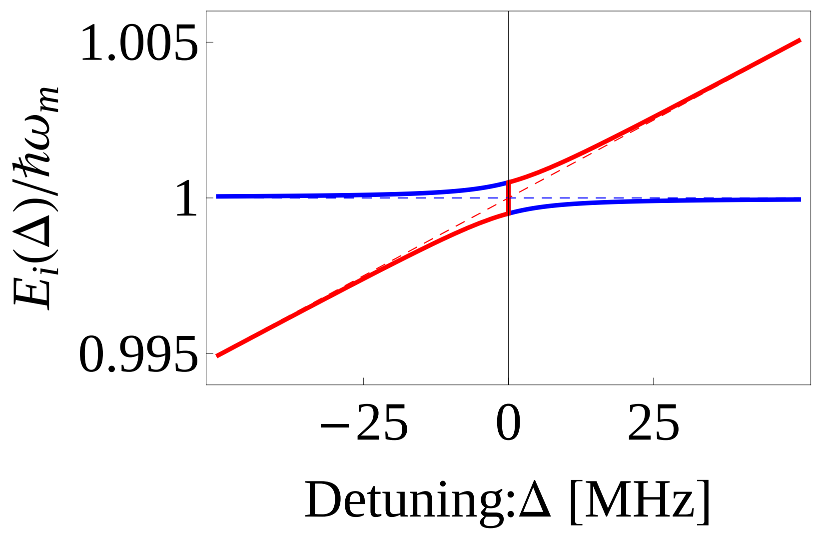

FIG. 4: Normalized eigen-energies $E_1(\Delta)$ and $E_2(\Delta)$ for the coupled oscillator system where the one oscillator has the bare energy of $\hbar\omega_m$ while the other has $\hbar\omega_m + \hbar\Delta$. Here, the coupling rate $\frac{g}{2\pi}$ is 10 MHz while $\frac{\omega}{2\pi}$ is 10 GHz. The dashed lines represent the eigen-energies for the uncoupled system, i.e., the dashed blue line is for the bare energy of mechanical oscillator and the dashed red line is for that of the LC circuit.

B. 1D atomic chain -phonon modes [4]

Let us consider one-dimensional monatomic atomic chain with the periodic (Born-von Karman) boundary condition [5], $q(N_\mathrm{a}a) = q(0)$, where a is the inter-atomic distance. The potential energy of an atom in the chain is now dependent on the configurations of the nearest-neighbor atoms. The total potential energy is then

$$V = \frac{1}{2}\kappa\sum_{n=1}^{N_\mathrm{a}}\left(q(na) - q([n+1]a)\right)^2, \tag{44}$$

where we assume that the neighboring atoms interact with the spring constant of κ. The equations of motion for the coordinates $\{q(a), \cdots, q(N_\mathrm{a}a)\}$ are coupled equations, i.e.,

$$\begin{aligned} m\ddot{q}(na) &= -\frac{\partial V}{\partial q(na)} \\ &= -\kappa\left(2q(na) - q([n-1]a) - q([n+1]a)\right). \end{aligned} \tag{45}$$

It is again possible to diagonalize the potential energy, Eq. (44) into independent quasi-particles' potential energies by linear transformation of the coordinates and thus by defining the normal modes (phonon modes). Taking advantage of the periodicity due to the periodic boundary condition ($q(N_\mathrm{a}a) = q(0)$) we can employ a type of Fourier transformation as the required transformation, that is,

$$q(na) = \frac{1}{\sqrt{N_\mathrm{a}}}\sum_{k_l} e^{ik_l na} u_{k_l}, \tag{46}$$

where $k_l = \frac{2\pi}{N_\mathrm{a}a} l$ with $l = 0, \pm 1, \pm 2, \cdots, \frac{N_\mathrm{a}}{2}$.

In terms of the normal coordinates u_{k_l}, the kinetic energy becomes

$$\begin{aligned} K &= \frac{1}{2N_\mathrm{a}}\sum_{na}\sum_{k_l}\sum_{k_{l'}} m\dot{u}_{k_l}\dot{u}_{k_{l'}} e^{i(k_l+k_{l'})na} \\ &= \frac{m}{2}\sum_{k_l}\dot{u}_{k_l}\dot{u}_{-k_l}, \end{aligned} \tag{47}$$

and the potential energy Eq. (44) becomes

$$\begin{aligned} V &= \frac{\kappa}{2N_\mathrm{a}}\sum_{na}\sum_{k_l}\sum_{k_{l'}} u_{k_l}u_{k_{l'}} e^{ik_l na}\left(e^{ik_l a} - 1\right) e^{ik'_l na}\left(e^{ik_{l'} a} - 1\right) \\ &= \frac{\kappa}{2}\sum_{k_l} 2\left(1 - \cos(k_l a)\right) u_{k_l}u_{-k_l}, \end{aligned} \tag{48}$$

where we used

$$\sum_{na} e^{i(k_l - k_{l'})na} = N_\mathrm{a}\delta_{k_l k_{l'}}. \tag{49}$$

Since the Lagrangian can be given by

$$L = \frac{m}{2}\sum_{k_l}\dot{u}_{k_l}\dot{u}_{-k_l} - \frac{\kappa}{2}\sum_{k_l} 2\left(1 - \cos(k_l a)\right) u_{k_l}u_{-k_l}, \tag{50}$$

the canonical momenta are systematically deduced, i.e.,

$$p_{k_l} = \frac{\partial L}{\partial \dot{u}_{k_l}} = m\dot{u}_{-k_l} \tag{51}$$

$$p_{-k_l} = \frac{\partial L}{\partial \dot{u}_{-k_l}} = m\dot{u}_{k_l}. \tag{52}$$

In terms of these we have the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2m}\sum_{k_l} p_{k_l}p_{-k_l} + \kappa\sum_{k_l}\left(1 - \cos(k_l a)\right) u_{k_l}u_{-k_l} \\ &= \sum_{k_l}\left(\frac{1}{2m}p_{k_l}p_{-k_l} + \frac{1}{2}m\frac{2\kappa\left(1 - \cos(k_l a)\right)}{m} u_{k_l}u_{-k_l}\right) \\ &= \sum_{k_l}\left(\frac{1}{2m}p_{k_l}p_{-k_l} + \frac{1}{2}m\omega_{k_l}^2 u_{k_l}u_{-k_l}\right), \end{aligned} \tag{53}$$

where in the last line ω_{k_l} is defined as

$$\omega_{k_l} = \sqrt{\frac{2\kappa\left(1-\cos(k_l a)\right)}{m}} = 2\sqrt{\frac{\kappa}{m}}\left|\sin(\frac{k_l a}{2})\right|. \tag{54}$$

Note that the relation between the eigen angular frequency ω (energy) and the wave number k (momentum) is generally called the *dispersion relation*. Figure 5 shows the dispersion for the case of an atomic chain consisted of 100 atoms. We can see that because of the inter-atomic interactions the eigen angular frequencies are *dispersed* from the nominal angular frequency of the individual oscillator ω_0.

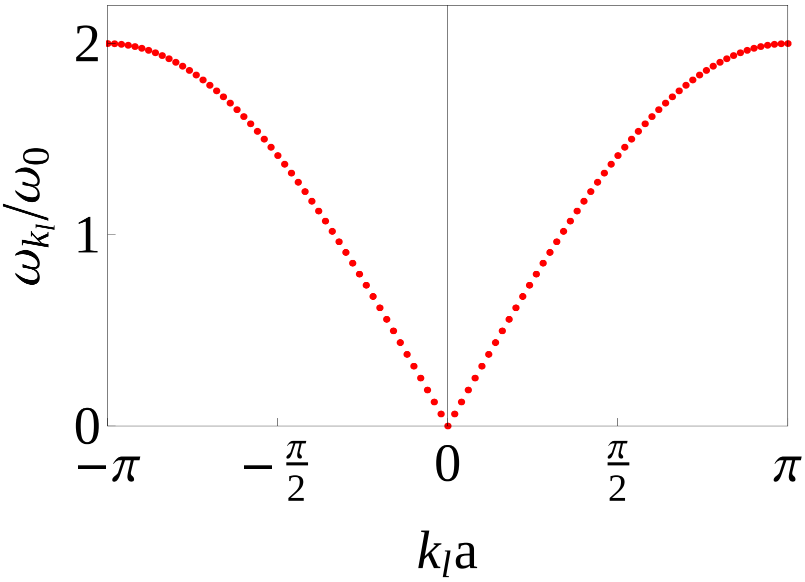

FIG. 5: Dispersion for the case of an atomic chain consisted of 100 atoms. The nominal angular frequency of the individual oscillator is ω_0 and $k_l a = \frac{2\pi}{100} l$, where $l = 0, \pm 1, \pm 2, \cdots, \pm 50$.

Now following Eqs. (10) and (11) the following annihilation and creation operators for the normal modes are defined

$$\hat{a}_{k_l} = \sqrt{\frac{m\omega_{k_l}}{2\hbar}}\left(\hat{u}_{-k_l} + \frac{i}{m\omega_{k_l}}\hat{p}_{k_l}\right) \tag{55}$$

$$\hat{a}^{\dagger}_{k_l} = \sqrt{\frac{m\omega_{k_l}}{2\hbar}}\left(\hat{u}_{k_l} - \frac{i}{m\omega_{k_l}}\hat{p}_{-k_l}\right). \tag{56}$$

Note that the annihilation and creation operators mix the modes specified by k_l and $-k_l$, which is a trait when dealing those operators with the periodic boundary condition [4]. With the annihilation and creation operators, Eqs. (55) and (56), the Hamiltonian (53) becomes

$$H = \sum_{k_l} \mathcal{H}_{k_l} \tag{57}$$

with the *Hamiltonian density*

$$\mathcal{H}_{k_l} = \hbar\omega_{k_l}\left(\hat{a}^{\dagger}_{k_l}\hat{a}_{k_l} + \frac{1}{2}\right). \tag{58}$$

III. BOSONIC FIELDS

Now let us see that from the coupled (0+1)-dimensional harmonic oscillators a (1+1)-dimensional Bosonic field is emerged by taking the *continuum limit*. As concrete examples we will deal with a longitudinal acoustic phonon mode in a 1D atomic chain in Sec. III A and an itinerant microwave photon mode propagating in a 1D transmission line in Sec. III B.

A. 1D atomic chain [4]

Suppose that the inter-atomic distance $a \ll 1$, then

$$q([n+1]a) - q(na) \equiv q(x_n + a) - q(x_n) = \frac{\partial q(x_n + a)}{\partial x} a \tag{59}$$

and

$$q([n]a) - q([n-1]a) \equiv q(x_n) - q(x_n - a) = \frac{\partial q(x_n)}{\partial x} a, \tag{60}$$

thus

$$\begin{aligned} 2q(na) - q([n-1]a) - q([n+1]a) &\equiv (q(x_n) - q(x_n - a)) - (q(x_n + a) - q(x_n)) \\ &= -\left(\frac{\partial q(x_n + a)}{\partial x} - \frac{\partial q(x_n)}{\partial x}\right) a \\ &= -\left(\frac{\partial^2 q(x_n + a)}{\partial x^2}\right) a^2. \end{aligned} \tag{61}$$

Plugging this in Eq. (45) the equation of motion for $q(x)$ in the *continuum limit* can be written as

$$m\ddot{q}(x) = \kappa \left(\frac{\partial^2 q(x)}{\partial x^2}\right) a^2, \tag{62}$$

or

$$\left(\frac{1}{v_s^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) q(x,t) = 0, \tag{63}$$

that is, a (1+1)-dimensional Bosonic field equation (massless Klein-Gordon equation) with the velocity,

$$v_s = \sqrt{\frac{\kappa a^2}{m}} = \sqrt{\frac{\kappa a}{\left(\frac{m}{a}\right)}} = \sqrt{\frac{c_{11}}{\rho}}, \tag{64}$$

where $\rho = \frac{m}{a}$ is the mass density of the chain and c_{11} is an elastic constant. This wave equation is essentially for the longitudinal acoustic phonon mode. Note that the dispersion relation Eq. (54) in the continuum limit is linear:

$$\omega_k = 2\sqrt{\frac{\kappa}{m}} \sin(\frac{ka}{2}) \rightarrow \sqrt{\frac{\kappa a^2}{m}} k = v_s k. \tag{65}$$

We see that in the continuum limit, $a \rightarrow 0, N_a \rightarrow \infty$, the displacement *field* $q(x,t)$ is emerged from the discrete atomic chain. Note that x is now the index as opposed to the coordinate. From the view point of the field theory the point mass equation, Eq. (6), is in fact (0+1)-dimensional boson field equation.

Note that the (0+1)-dimensional coordinates have been defined by Eq. (46) as

$$q_n(t) = \frac{1}{\sqrt{N_a}} \sum_{k_l} e^{ik_l(na)} u_{k_l}(t), \tag{66}$$

and the canonical momentum is

$$p_n(t) = \frac{1}{\sqrt{N_a}} \sum_{k_l} e^{-ik_l(na)} p_{k_l}(t). \tag{67}$$

Thus the commutation relations are

$$[q_n(t), p_{n'}(t)] = i\hbar\delta_{n,n'}. \tag{68}$$

for the real space operators since we have

$$\left[u_{k_l}(t), p_{k_{l'}}(t)\right] = i\hbar\delta_{k_l,k_{l'}} \tag{69}$$

for the reciprocal space operators.

The (1+1)-dimensional operators are defined, on the other hand, by

$$q(x,t)=\lim_{\substack{a\to 0\\ N_a\to\infty}}\frac{q_n(t)}{\sqrt{a}}=\lim_{\substack{a\to 0\\ N_a\to\infty}}\frac{1}{\sqrt{N_a a}}\sum_k u_k(t)e^{ikx}=\frac{1}{\sqrt{L}}\sum_k u_k(t)e^{ikx} \tag{70}$$

and

$$p(x,t)=\lim_{\substack{a\to 0\\ N_a\to\infty}}\frac{p_n(t)}{\sqrt{a}}=\lim_{\substack{a\to 0\\ N_a\to\infty}}\frac{1}{\sqrt{N_a a}}\sum_k p_k(t)e^{-ikx}=\frac{1}{\sqrt{L}}\sum_k p_k(t)e^{-ikx}, \tag{71}$$

where the factor $\frac{1}{\sqrt{a}}$ is introduced in each definition in order to make sense when the limit operation $a\to 0, N_a\to\infty$ with $N_a a = L$ finite is performed on the original (0+1)-dimensional forms in Eqs. (66) and (67). The Fourier-transforms are then properly defined with the periodic boundary condition over the length L

$$u_k(t)=\frac{1}{\sqrt{L}}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx q(x,t)e^{-ikx} \tag{72}$$

and

$$p_k(t)=\frac{1}{\sqrt{L}}\int_{-\frac{L}{2}}^{\frac{L}{2}} dx p(x,t)e^{ikx}. \tag{73}$$

The commutation relations are then

$$[q(x,t),p(x',t)]=i\hbar\delta(x-x') \tag{74}$$

for the real space operators and

$$[u_k,p_{k'}]=i\hbar\delta_{k,k'} \tag{75}$$

for the reciprocal space operators [4].

B. Transmission line [6]

The language we have developed for treating a longitudinal acoustic phonon mode in a 1D atomic chain as a Bosonic field can be translated into that for an itinerant microwave photon mode propagating in a 1D transmission line (see the correspondences table, Table III B 1). The field equation of the transmission line can be read as

$$\left(\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\varphi(x,t)=0, \tag{76}$$

where

$$v_p=\frac{1}{\sqrt{lc}} \tag{77}$$

with l being the inductance per unit length and c being the capacitance to the ground per unit length.

1. *Lagrangian and Hamiltonian formalism*

The heuristic approach presented above can be justified by the following arguments. The *flux variable* $\varphi(x,t)$ in Eq. (76) is to the 1D transmission line what the displacement field $q(x,t)$ is to the 1D atomic chain. The flux variable $\varphi(x,t)$ here is related to the *local voltage* $V(x,t)$ as [3]

$$\varphi(x,t)=\int_{-\infty}^{t} d\tau V(x,\tau), \tag{78}$$

which leads to Faraday's law of induction:

$$V(x,t) = \dot{\varphi}(x,t). \tag{79}$$

On the other hand, the voltage drop $\delta V(x,t)$ within a length δx can be expressed in terms of the local magnetic field, $\boldsymbol{B}(x,t)$, as

$$\delta V(x,t) = -\frac{\partial}{\partial t} \underbrace{\delta \iint_{\Sigma} \boldsymbol{B}(x,t) \cdot d\boldsymbol{S}}_{l\delta x I(x,t)} = -l\delta x \frac{\partial}{\partial t} I(x,t), \tag{80}$$

and thus

$$\frac{\partial}{\partial x} V(x,t) = -l \frac{\partial}{\partial t} I(x,t), \tag{81}$$

where $I(x,t)$ is the *local current*. With Eqs. (79) and (81) we have

$$\left(\frac{\partial}{\partial t} I(x,t)\right) = -\frac{1}{l}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\varphi(x,t)\right), \tag{82}$$

thus,

$$I(x,t) = -\frac{1}{l}\frac{\partial}{\partial x}\varphi(x,t). \tag{83}$$

With these relations (79) and (83) the Lagrangian density can be written as

$$\begin{aligned}
\mathcal{L}(x,t) &= \frac{c}{2} V(x,t)^2 - \frac{l}{2} I(x,t)^2 \\
&= \underbrace{\frac{c}{2}\left(\dot{\varphi}(x,t)\right)^2}_{\text{Kinetic energy}} - \underbrace{\frac{l}{2}\left(-\frac{1}{l}\frac{\partial}{\partial x}\varphi(x,t)\right)^2}_{\text{Potential energy}} \\
&= \frac{c}{2}\left(\dot{\varphi}(x,t)\right)^2 - \frac{1}{2l}\left(\frac{\partial}{\partial x}\varphi(x,t)\right)^2.
\end{aligned} \tag{84}$$

From this Lagrangian density Eq. (84) the Euler-Lagrange equation can be deduced [7] as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} + \underbrace{\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \frac{\partial}{\partial x}\varphi}}_{\text{extra term}} = 0, \tag{85}$$

which is exactly the wave equation we had in Eq. (76).

We can confirm that the momentum conjugate to $\varphi(x,t)$ is indeed the charge density,

$$q(x,t) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = cV(x,t). \tag{86}$$

2. *Canonical quantization*

The Hamiltonian density is

$$\mathcal{H}(x) = \frac{1}{2c} q(x,t)^2 + \frac{1}{2l}\left(\frac{\partial}{\partial x}\varphi(x,t)\right)^2 \tag{87}$$

and the Hamiltonian is obtained by integrating the Hamiltonian density Eq. (87) over the length L:

$$H = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \mathcal{H}(x). \tag{88}$$

The commutation relation is

$$[\varphi(x,t), q(x',t)] = i\hbar\delta(x-x') \tag{89}$$

for the real space operators. Note here that Dirac' delta function (as opposed to Kronecker's delta function) apears as a result of taking the continuum limit, that is, the inter-LC distance $a \to 0$ and the number of LC $N_a \to \infty$.

In reciprocal space the flux and charge variables are defined with the aforementioned periodic boundary condition as

$$\varphi_k = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) e^{ikx} \tag{90}$$

$$q_k = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx q(x) e^{-ikx}. \tag{91}$$

Here the commutation relation is still with Kronecker's delta function,

$$[\varphi_k, q_{k'}] = i\hbar\delta_{kk'}. \tag{92}$$

The Hamiltonian can be rewritten in terms of φ_k and q_k as

$$H = \sum_k \mathcal{H}_k \tag{93}$$

with

$$\begin{aligned} \mathcal{H}_k &= \frac{1}{2c} q_k q_{-k} + \frac{1}{2l} \varphi_k \varphi_{-k} \\ &= \hbar\omega_k \left(\hat{c}_k^\dagger \hat{c}_k + \frac{1}{2} \right), \end{aligned} \tag{94}$$

where the annihilation and creation operators for the Bosonic field are respectively defined by

$$\hat{c}_k = \sqrt{\frac{c\omega_n}{2\hbar}} \left(\varphi_{-k} + \frac{i}{c\omega_k} q_k \right) \tag{95}$$

$$\hat{c}_k^\dagger = \sqrt{\frac{c\omega_l}{2\hbar}} \left(\varphi_k - \frac{i}{c\omega_k} q_{-k} \right), \tag{96}$$

with the commutation relation

$$[\hat{c}_k, \hat{c}_{k'}] = \delta_{kk'} \tag{97}$$

These are the same as the forms for the φ-representation of a LC circuit except for the quantities φ_k and q_k are complex variable due to the fact that the periodic boundary condition is used to formulate.

TABLE II: Atomic chain - Transmission line - EM traveling mode

	Atomic chain	Transmission line	EM traveling mode
Mass density	ρ	c	ϵ_0
Elastic const.	c_{11}	$\frac{1}{l}$	$\frac{1}{\mu_0}$
Velocity	$v_s = \sqrt{\frac{c_{11}}{\rho}}$	$v_p = \frac{1}{\sqrt{cl}}$	$c_v = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$
Impedance	$Z_s = \sqrt{\frac{1}{\rho c_{11}}}$	$Z_p = \sqrt{\frac{l}{c}}$	$Z_v = \sqrt{\frac{\mu_0}{\epsilon_0}}$
Displacement	$q(x,t)$	$\varphi(x,t)$	$A(\omega)$
Momentum	$p(x,t)$	$q(x,t)$	$\Pi(\omega)$

3. Second continuum limit (thermodynamic limit)

Taking the *second* continuum limit (thermodynamic limit) $L \to \infty$ makes the sum on k in Eq. (93) changed into the integral over k:

$$H = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hbar\omega_k \left(\hat{c}^\dagger(k)\hat{c}(k) + \frac{1}{2} \right), \tag{98}$$

where

$$\hat{c}(k) = \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(-k) + \frac{i}{c\omega_k} q(k) \right) \tag{99}$$

$$\hat{c}^\dagger(k) = \sqrt{\frac{c\omega_k}{2\hbar}} \left(\varphi(k) - \frac{i}{c\omega_k} q(-k) \right), \tag{100}$$

with

$$\varphi(k) = \lim_{L\to\infty} \sqrt{L}\varphi_k = \int_{-\infty}^{\infty} dx \varphi(x) e^{ikx} \tag{101}$$

$$q(k) = \lim_{L\to\infty} \sqrt{L} q_k = \int_{-\infty}^{\infty} dx q(x) e^{-ikx}. \tag{102}$$

The commutation relation for the canonical operators becomes

$$\begin{aligned} [\varphi(k), q(k')] &= \left[\int_{-\infty}^{\infty} dx \varphi(x) e^{ikx}, \int_{-\infty}^{\infty} dx' q(x') e^{-ik'x'} \right] \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \underbrace{[\varphi(x), q(x')]}_{i\hbar\delta(x-x')} e^{i(kx-k'x')} \\ &= i\hbar \underbrace{\int_{-\infty}^{\infty} dx e^{i(k-k')x}}_{2\pi\delta(k-k')} \\ &= i\hbar \, 2\pi\delta(k-k'), \end{aligned} \tag{103}$$

and thus that for the annihilation and creation operators is

$$[\hat{c}(k), \hat{c}^\dagger(k)] = 2\pi\delta(k-k') \tag{104}$$

from Eqs. (99), (100), and (103).

Appendix A: Boundary conditions

We shall summarize popular boundary conditions used in the literature to help clarify the differences. These are particularly relevant to the situation where the quantization is performed on the finite size system, that is, when making do without the second continuum limit (thermodynamic limit).

Periodic (Born-von Karman) boundary condition (Traveling wave [*complex*])

- Boundary condition: $\varphi(0) = \varphi(L)$
- Variable: $\varphi(x) = \frac{1}{\sqrt{L}} \sum_{k_n} \varphi_n e^{ik_n x}$ [N terms]
- Fourier transform: $\varphi_n = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) e^{-ik_n x}$
- $k_n = \frac{2\pi}{L} n$

- $n = \underbrace{0, \pm 1, \pm 2, \cdots, \pm \frac{N}{2} - 1, \frac{N}{2}}_{\text{N points}}$

Dirichlet boundary condition (Fixed-end standing wave [*real*])

- Boundary condition: $\varphi(0) = \varphi(L) = 0$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum_{k_n \geq 0} \varphi_n^{(s)} \sin(k_n x)$ [N terms]
- Fourier transform: $\varphi_n^{(s)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \sin(k_n x)$
- $k_n = \frac{\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, N-1, N}_{\text{N points}}$

Neumann boundary condition (Open-end standing wave [*real*])

- Boundary condition: $\frac{\partial}{\partial x}\varphi(0) = \frac{\partial}{\partial x}\varphi(L) = 0$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum_{k_n \geq 0} \varphi_n^{(c)} \cos(k_n x)$ [N terms]
- Fourier transform: $\varphi_n^{(c)} = \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \cos(k_n x)$
- $k_n = \frac{\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, N-1, N}_{\text{N points}}$

Mixed boundary condition (Cosine and sine modes [*real*])

- Boundary condition: $\varphi(0) = \varphi(L)$
- Variable: $\varphi(x) = \sqrt{\frac{2}{L}} \sum_{k_n \geq 0} \left(\varphi_n^{(c)} \cos(k_n x) - \varphi_n^{(s)} \sin(k_n x) \right)$ [$\frac{N}{2}$ terms $\times$ 2 modes (cosine and sine)]
- Fourier transform: $\begin{aligned} \varphi_n^{(c)} &= \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \cos(k_n x) \\ \varphi_n^{(s)} &= \sqrt{\frac{2}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x) \sin(k_n x) \end{aligned}$
- $k_n = \frac{2\pi}{L} n$
- $n = \underbrace{0, 1, 2, \cdots, \frac{N}{2} - 1}_{\text{N/2 points}}$

We will henceforth chiefly employ periodic (Born-von Karman) boundary condition.

Appendix B: Remarks on dimensions

There are many confusing points we have to be careful in dealing with Bosonic fields. Among other things I would like to emphasize in particular the issue of dimensions: we have to be aware that the dimensions of the operators change when the continuum limits are taken. Here we see these dimensional changes by taking a 1D transmission line as an example.

a. Discrete (0+1)-dimensional variables (N coupled-LC chain)

The cannonical variables are

$$\varphi_n(t) = \frac{1}{\sqrt{N}} \sum_{k_l} e^{-ik_l(na)} \varphi_{k_l}(t) \tag{B1}$$

$$q_n(t) = \frac{1}{\sqrt{N}} \sum_{k_l} e^{ik_l(na)} q_{k_l}(t), \tag{B2}$$

where

$$k_l = \frac{2\pi}{Na} l \tag{B3}$$

with $l = 0, \pm 1, \pm 2, \cdots, \pm \frac{N}{2}$. The commutation relations are given by

$$[\varphi_n(t), q_{n'}(t)] = i\hbar \delta_{n,n'}. \tag{B4}$$

The Fourier transforms read

$$\varphi_{k_l}(t) = \frac{1}{\sqrt{N}} \sum_{n} e^{ik_l(na)} \varphi_n(t) \tag{B5}$$

$$q_{k_l}(t) = \frac{1}{\sqrt{N}} \sum_{n} e^{-ik_l(na)} q_n(t), \tag{B6}$$

and the commutation relations

$$\left[\varphi_{k_l}(t), q_{k_{l'}}(t)\right] = i\hbar \delta_{k_l, k_{l'}}. \tag{B7}$$

b. Continuum (1+1)-dimensional variables (length-L 1D transmission line)

By taking a *first* continuum limit, (1+1)-dimensional operators are defined by

$$\varphi(x,t) \equiv \lim_{\substack{a\to 0\\ N\to\infty}} \frac{\varphi_n(t)}{\sqrt{a}} = \lim_{\substack{a\to 0\\ N\to\infty}} \frac{1}{\sqrt{Na}} \sum_k \varphi_k(t) e^{-ikx} = \frac{1}{\sqrt{L}} \sum_k \varphi_k(t) e^{-ikx} \tag{B8}$$

$$q(x,t) \equiv \lim_{\substack{a\to 0\\ N\to\infty}} \frac{q_n(t)}{\sqrt{a}} = \lim_{\substack{a\to 0\\ N\to\infty}} \frac{1}{\sqrt{Na}} \sum_k q_k(t) e^{ikx} = \frac{1}{\sqrt{L}} \sum_k q_k(t) e^{ikx}, \tag{B9}$$

where the factor $\frac{1}{\sqrt{a}}$ is introduced in each definition in order to make sense when the limit operation $a \to 0, N \to \infty$ with $Na = L$ being finite is performed. The Fourier-transforms are then properly defined with the periodic boundary condition over the length L

$$\varphi_k(t) = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \varphi(x,t) e^{ikx} \tag{B10}$$

$$q_k(t) = \frac{1}{\sqrt{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx q(x,t) e^{-ikx}. \tag{B11}$$

For the real space operators the commutation relations are now given with *Dirac's delta*

$$[\varphi(x,t), q(x',t)] = i\hbar \delta(x - x') \tag{B12}$$

while for the reciprocal space operators these are still given with *Kronecker's delta*

$$[\varphi_k, q_{k'}] = i\hbar \delta_{k,k'}. \tag{B13}$$

c. *Continuum (1+1)-dimensional variables (infinite-length 1D transmission line)*

Taking the *second* continuum limit (thermodynamic limit), $L \to \infty$, we have

$$\varphi(k) = \lim_{L\to\infty} \sqrt{L}\varphi_k = \int_{-\infty}^{\infty} dx\varphi(x)e^{ikx} \tag{B14}$$

$$q(k) = \lim_{L\to\infty} \sqrt{L}q_k = \int_{-\infty}^{\infty} dxq(x)e^{-ikx}. \tag{B15}$$

These changes in turn lead the following representations of real space operators:

$$\varphi(x,t) = \lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_k \frac{\varphi(k)}{\sqrt{L}} e^{-ikx} = \lim_{L\to\infty} \frac{1}{L} \sum_k \varphi(k)e^{-ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\varphi(k)e^{-ikx} \tag{B16}$$

$$q(x,t) = \lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_k \frac{q(k)}{\sqrt{L}} e^{ikx} = \lim_{L\to\infty} \frac{1}{L} \sum_k q(k)e^{ikx} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}q(k)e^{ikx}, \tag{B17}$$

The commutation relations for the canonical operators become

$$[\varphi(k), q(k')] = i\hbar\ 2\pi\delta(k-k') \tag{B18}$$

and

$$[\varphi(x,t), q(x',t)] = i\hbar\delta(x-x'). \tag{B19}$$

Appendix C: (3+1)-dimensional Bosonic field

We have learned (1+1)-dimensional Bosonic field and used to analyze a longitudinal acoustic phonon mode in a 1D atomic chain in Sec. III A and a microwave photon mode propagating in a 1D transmission line in Sec. III B. Now let us extend the spatial dimension from 1 to 3 to learn a bit about (3+1)-dimensional Bosonic field.

1. Elastic waves in an isotropic medium [4, 8]

For fluids the volume v is the only relevant mechanical extensive parameter. For solid, however, there are several additional mechanical extensive parameters. The emergence of these additional parameters can be said as a consequence of the "spontaneous symmetry breaking" occurring in the ordered systems. The relevant extensive parameters are the strain tensor components, which can be written as the elements of a symmetric matrix;

$$\mathbf{\Sigma} = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{xy} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{xz} & \Sigma_{yz} & \Sigma_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x(\boldsymbol{r})}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x(\boldsymbol{r})}{\partial y} + \frac{\partial u_y(\boldsymbol{r})}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x(\boldsymbol{r})}{\partial z} + \frac{\partial u_z(\boldsymbol{r})}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x(\boldsymbol{r})}{\partial y} + \frac{\partial u_y(\boldsymbol{r})}{\partial x}\right) & \frac{\partial u_y(\boldsymbol{r})}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y(\boldsymbol{r})}{\partial z} + \frac{\partial u_z(\boldsymbol{r})}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_x(\boldsymbol{r})}{\partial z} + \frac{\partial u_z(\boldsymbol{r})}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y(\boldsymbol{r})}{\partial z} + \frac{\partial u_z(\boldsymbol{r})}{\partial y}\right) & \frac{\partial u_z(\boldsymbol{r})}{\partial z}, \end{bmatrix} \tag{C1}$$

with $\boldsymbol{u}(\boldsymbol{r}) = \{u_x(\boldsymbol{r}), u_y(\boldsymbol{r}), u_z(\boldsymbol{r})\}$ being the displacement field vector at position $\boldsymbol{r}$. Note that with the displacement field vector the infinitesimal distance $d\boldsymbol{r}$ in the elastic body can be deformed as $d\boldsymbol{r}' = d\boldsymbol{r} + d\boldsymbol{r}\cdot\nabla\boldsymbol{u}(\boldsymbol{r})$ and the strain tensor $\mathbf{\Sigma}$ is nothing but the symmetric part of $\nabla\boldsymbol{u}(\boldsymbol{r})$.

The cubic symmetry is the highest form of symmetry of any crystalline solid can have. A non-crystalline amorphous solid, however, can be isotropic. Since an isotropic system is invariant under any rotation we have just 2 independent elastic coefficients, λ_{L} and μ_{L}, called the Lamé constants. In this case the free elastic energy has to be expressed as a manifestly rotational invariant form with λ_{L} and μ_{L}:

$$F = \frac{1}{2}\lambda_{\mathrm{L}}\left(\sum_\alpha \Sigma_{\alpha\alpha}\right)^2 + \mu_{\mathrm{L}}\sum_\alpha\sum_\beta \Sigma_{\alpha\beta}^2, \tag{C2}$$

where $\alpha = \{x,y,z\}$ and $\beta = \{x,y,z\}$. Note that the first term, $\left(\sum_\alpha \Sigma_{\alpha\alpha}\right)^2$, is the *squared sum* of the diagonal elements of $\mathbf{\Sigma}$ and the second term, $\sum_\alpha\sum_\beta \Sigma_{\alpha\beta}^2$, is the *sum of squares* of all elements of $\mathbf{\Sigma}$.

The Lagrangian density for the isotropic elastic medium can then be given by

$$\mathcal{L}(u_\alpha, \cdots, \dot{u}_\alpha, \cdots) = \frac{1}{2}\sum_\alpha \rho \dot{u}_\alpha^2 - \frac{1}{2}\lambda_{\rm L}\left(\sum_\alpha \Sigma_{\alpha\alpha}\right)^2 - \mu_{\rm L}\sum_\alpha\sum_\beta \Sigma_{\alpha\beta}^2. \tag{C3}$$

Using the Euler-Lagrange equation (85) the following elastic wave equations can be deduced

$$\left(\frac{1}{v_L^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u_L(x,t) \;=\; 0 \tag{C4}$$

$$\left(\frac{1}{v_T^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) u_T(x,t) \;=\; 0, \tag{C5}$$

where the former represents the longitudinal acoustic phonon mode with the longitudinal sound velocity

$$v_L = \sqrt{\frac{\lambda_{\rm L} + 2\mu_{\rm L}}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu_{\rm L}}{\rho}} = \sqrt{\frac{c_{11}}{\rho}}, \tag{C6}$$

while the latter represents the transverse acoustic phonon mode with the transverse sound velocity

$$v_T = \sqrt{\frac{\mu_{\rm L}}{\rho}} = \sqrt{\frac{c_{44}}{\rho}}. \tag{C7}$$

Here K is the bulk modulus, which is written as $K = \lambda + \frac{2}{3}\mu_{\rm L}$. The Lamé constant $\mu_{\rm L}$ is also called the shear modulus.

2. Electromagnetic waves in vacuum [7]

The electomagnetic waves can be similarly obtained when the Lorentz invariance is forgone and Coulomb guage is used. Formally replacing the displacement field $u_\alpha(\boldsymbol{r})$ with the vector potential $A_\alpha(\boldsymbol{r})$, ρ with ϵ_0, and $\mu_{\rm L}$ with $\frac{1}{\mu_0}$ as suggested in Table III B 1 we shall have electromagnetic wave equations. In Coulomb guage,

$$\nabla \cdot \boldsymbol{A} = 0, \tag{C8}$$

we only have the transverse mode and the wave equation can be read as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) A_T(x,t) = 0. \tag{C9}$$

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