Unstable states and bounces

Koji Usami*
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In this final lecture, we shall use the instanton methods to deal with the *decay processes* of unstable states, i.e., *bounces*. This is the basis of the celebrated *Caldeira-Leggett model* of quantum tunneling in macroscopic systesms.

I. DIALOGUE CONCERNING THE UNSTABLE SYSTEMS

What follows are almost the verbatim transcriptions of the legendary article called "The uses of instantons" by Sidney Coleman [1]:

Sagredo: Let me test my understanding of these instanton methods by studying a potential

$$V(q) = -\frac{k}{2\sigma} q^2 (q - \sigma), \tag{1}$$

which is shown in Fig. 1. If I neglect barrier penetration, in the semiclassical limit, this potential has an energy eigenstate sitting in the bottom of the well. I wish to compute the corrections to the energy of this state due to barrier penetration. If I turn the potential upside down (Fig. 2), I observe that the classical equation of motion has a solution in which the particle begins at the top of the hill at $q = 0$, bounces off the classical turning point σ, and returns to the top of the hill as shown in Fig. 3. I will call this motion 'the bounce'. I will compute the transition matrix elemet between $q = 0$ and $q = 0$ by summing over configurations consisting of widely separated bounces, just as one sums over instantons and anti-instantons in the study of the double well. Indeed, the sum is the same as that for the double well (with the obvious redefinitions of S_{in} ($S_{in} \to S_b$), ω, etc.), save that there is no restriction to an even or odd number of bounces. Thus I obtain the complete exponential series, rather than just the odd or even terms, and I find that

$$\langle q = 0| \exp\left[-\frac{\tau}{\hbar} H\right] |q = 0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{\omega\tau}{2}} \exp\left[\tau K e^{-\frac{S_b}{\hbar}}\right], \tag{2}$$

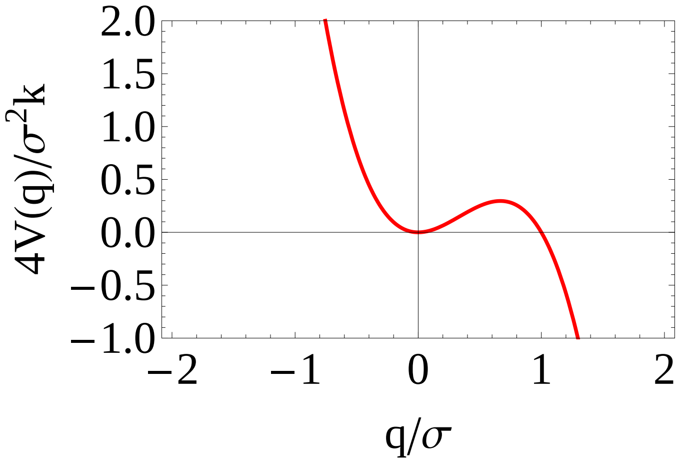

FIG. 1: A potential.

*Electronic address: usami@qc.rcast.u-tokyo.ac.jp

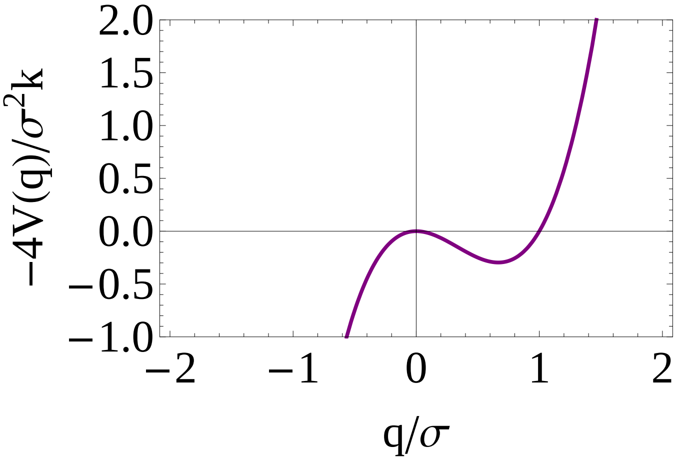

FIG. 2: An inverted potential.

and the energy eigenvalue is given by

$$\epsilon_0 = \frac{\hbar\omega}{2} - \hbar K e^{-\frac{S_b}{\hbar}}. \tag{3}$$

Salviati: Alas, Sagredo, I fear you have erred in three ways. Firstly, the term you have computed is small compared to terms of order $\hbar^2$ which you have neglected, and thus you have no right to retain it. Secondly, I see by your sketch shown in Fig. 3 that the bounce has a maximum; therefore the zero-mode eigenfunction

$$x_1 = \sqrt{\frac{m}{S_b}}\frac{dq}{d\tau}, \tag{4}$$

which is proportional to the time derivative of the bounce, has a node as shown in Fig. 4. Thus it is not the eigenfunction of lowest eigenvalue, and there must be a nodeless eigenfunction, x_0, of a lower eigenvalue, that is to say, there must be a negative eigenvalue. Thus K, which is inversely proportional to the product of the square roots of the eigenvalues, is imaginary. Thirdly, the eigenvalue you attempt to compute is nowhere to be found in the spectrum of the Hamiltonian, because the state you are studying is rendered unstable by the barrier penetration.

Sagredo: Everything you say is correct, but I believe your criticisms show how to save the computation. An unstable state is one whose energy has an imaginary part; thus it is only to be expected that K should be imaginary.

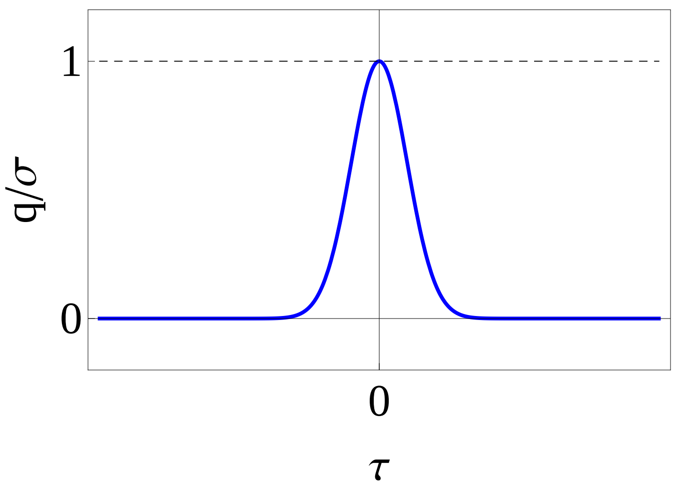

FIG. 3: A trajectory of a bounce.

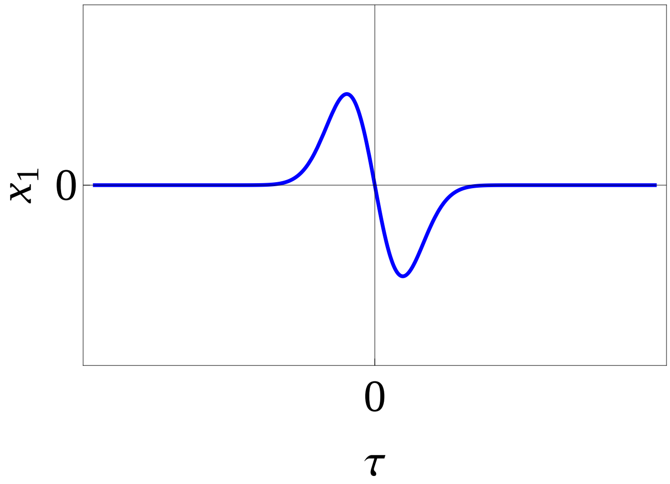

FIG. 4: Eigenfunction of the zero mode.

Furthermore, the term I have computed, though indeed small compared to neglected contributions to the real part of ϵ_0, is the leading contribution to the imaginary part of ϵ_0. Thus the correct version of Eq. (3) is

$$\mathrm{Im}\left[\epsilon_0\right] = \frac{\Gamma}{2} = \hbar\,|K|\,e^{-\frac{S_b}{\hbar}} \tag{5}$$

where Γ is, as usual, the width of the unstable state.

The above argument is quite amazing [2]. If we consider only small fluctuations arond $q = 0$ and neglecting the contributions of bounces we only find

$$\langle q=0|\exp\left[-\frac{\tau}{\hbar}H\right]|q=0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{\omega\tau}{2}}. \tag{6}$$

By Wick rotation again, i.e., $\tau \to it$, we have

$$\langle q=0|\exp\left[-i\frac{H}{\hbar}t\right]|q=0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}e^{-i\frac{\omega}{2}t}, \tag{7}$$

and there is no decay. Thus, we find that the quantum tunnelings, that is, the bounces, bring about the decay for the unstable system!

II. ANALYTIC CONTINUATION

Coleman said, however, that [1]:

Coleman: Sagredo has missed a factor of $\frac{1}{2}$. The correct answer is

$$\Gamma = \hbar\,|K|\,e^{-\frac{S_b}{\hbar}}. \tag{8}$$

To show that this is the case requires a more careful argument than Sagredo's. The essential point is Salviati's observation that the energy of an unstable state is not an eigenvalue of H; in fact, it is an object that can only be defined by a precess of *analytic continuation*.

To keep things as simple as possible, let us consider not an integral over all function space, but an intergral over some path in function space parameterized by a real variable, z,

$$J = \int \frac{dz}{\sqrt{2\pi\hbar}}e^{-\frac{S(z)}{\hbar}}, \tag{9}$$

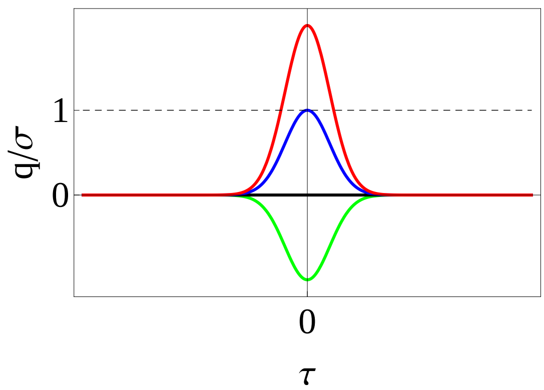

FIG. 5: A Path in function space parameterizedd by a real variable z. Green function: $z = -1$, Black function: $z = 0$, Blue function: $z = 1$, and Red function: $z = 2$.

where $S(z)$ is the action along the path. In particular, let us choose the path sketched in Fig. 5. This path includes two important functions that occur in the real problem: $q(\tau) = 0$, at $z = 0$, and the bounce, at $z = 1$. Furthermore, the path is such that the tangent vector to the path at $z = 1$ is x_1 given in Eq. (4). Thus the path goes through the bounce in the 'most dangerous direction', that direction with which the negative eigenvalue is associated, and $z = 1$ is a maximum of $S(z)$, as shown in Fig. 6. $S(z)$ goes to minus infinity as z goes to infinity because the functions spend more and more time in the region beyond the turning point, where V is negative; note that this implies that Eq. (9) is hopelessly divergent.

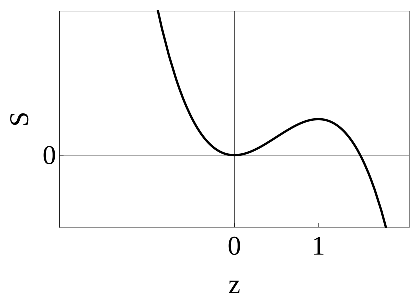

FIG. 6: Action S with respect to z.

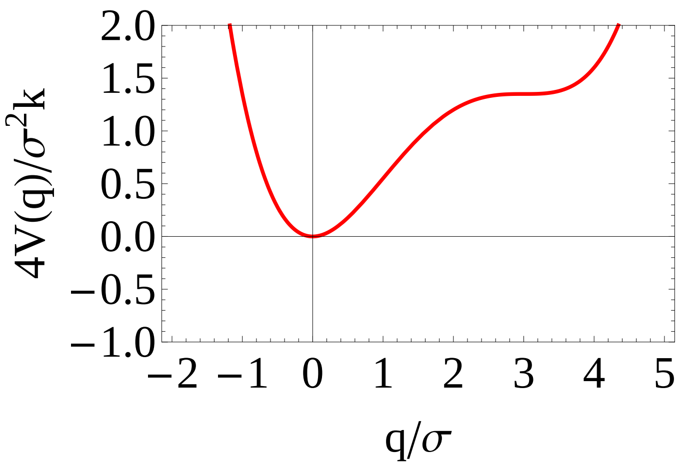

FIG. 7: Analytically changed potential.

If $q = 0$ were the absolute minimum of V, that is to say, if V were as shown in Fig. 7, we would have, for the same

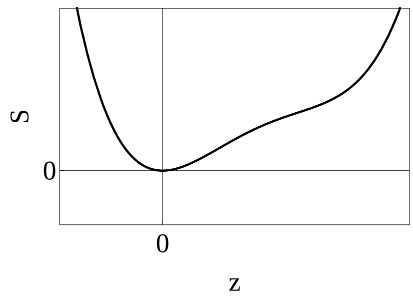

FIG. 8: Resultant action S with respect to z.

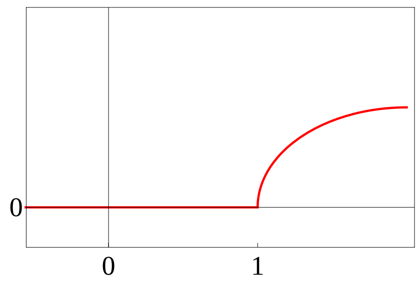

FIG. 9: Complex plane z. The contour of the integration in Eq. (9) is distorted into the upper half plane at $z = 1$, the saddle point.

path, the situation shown in Fig. 8, and there would be no divergence in Eq. (9). Now let us suppose we analytically chage V in some way such that we go from this situation back to the one of interest. To keep the integral convergent, we must distort the right-hand portion of the contour of integration into the complex plane. How we distort it depends on the details of the analytic passage from one potential to the other. In Fig. 9 I have assumed that it is distirted into the upper half plane. Following the standard procedure of the method of steepest descents, I have led the contour along the real axis to $z = 1$, the saddle point, and then out along a line of constant imaginary part of $S(z)$. The integral thus acquires an imaginary part; in the steepest-descent approxiamtion,

$$\begin{aligned} \mathrm{Im}\,[J] &= \mathrm{Im}\left[\int_1^{1+i\infty} \frac{dz}{\sqrt{2\pi\hbar}} e^{-\frac{S(1)}{\hbar}} e^{-\frac{1}{2}\frac{S''(1)}{\hbar}(z-1)^2}\right] \\ &= \frac{1}{2} e^{-\frac{S(1)}{\hbar}} \frac{1}{\sqrt{|S''(1)|}}. \end{aligned} \tag{10}$$

Note the factor of $\frac{1}{2}$; this arises because the integration is over only half of the Gaussian peak.

Now, we have studied a one-dimensional integral, but we can alwayways reduce our functional integral to a one-dimensional integral simply by integrating (in Gaussian approximation) over all the variables orthogonal to our path. These directions involve only positive or zero eigenvalues near the stationary point and give us no trouble. In this manner we obtain Sagredo's answer, Eq. (5), except that the negative eigenvalue carries a factor of $\frac{1}{2}$ with it; that is to say, we obtain Eq. (8).

[1] Sidney Coleman, *Aspect of Symmetry*, (Cambridge University Press, Cambridge 1985), Chapter 7.
[2] Xiao-Gang Wen, *Quantum Field Theory of Many-Body Systems*, (Oxford University Press, Oxford, 2004).