Report Problems

Koji Usami*
(Dated: December 10, 2018)

Through these problems, we explore the *quantum Hall effect* for a 2D noninteracting electrons.

I. ANOMALOUS VELOCITY

Let us consider a non-interacting elecrron in a 2D crystal laying xy-plane under the perturbation of a weak and uniform electric field $\boldsymbol{E}$ also laying xy-plane. The uniform field $\boldsymbol{E}$ leads to the linearly changing electrostatic potential $\phi(x, y)$, which breaks the translational symmetry of the crystal. To exploit the Bloch's theorem for the original Hamiltonian without electric field $\boldsymbol{E}$,

$$H_0 = \frac{1}{2m}\boldsymbol{p}^2 + V(x, y), \tag{1}$$

the electric field can be thought of appearing through the *uniform* but *time-dependent* vector potential $\boldsymbol{A}(t)$ with

$$\boldsymbol{E} = -\dot{\boldsymbol{A}}(t). \tag{2}$$

Then the Hamiltonian (1) is modified into

$$H(t) = \frac{1}{2m}\left(\boldsymbol{p} + e\boldsymbol{A}(t)\right)^2 + V(x, y), \tag{3}$$

where m is the mass, e is the charge, $\boldsymbol{p}$ is the canonical momentum of the electron. Here, $V(\boldsymbol{r})$ is the periodic potential for the electron created by the crystal.

Since the original Hamiltonian (1) possesses the translational symmetry, introducing the crystal momentum $\boldsymbol{q}$ the instantaneous eigenstates for H_0 can be given by the Bloch form

$$|\psi_{n,\boldsymbol{q}}(\boldsymbol{r})\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{r}}|u_{n,\boldsymbol{q}}(\boldsymbol{r})\rangle. \tag{4}$$

Now further introducing the *gauge-invariant* crystal momentum,

$$\boldsymbol{k} = \boldsymbol{q} + \frac{e}{\hbar}\boldsymbol{A}(t) \tag{5}$$

the instantaneous eigenstates for $H(t)$ can be similarly given by the Bloch form

$$|\psi_{n,\boldsymbol{k}}(\boldsymbol{r})\rangle = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|u_{n,\boldsymbol{k}}(\boldsymbol{r})\rangle. \tag{6}$$

Since $\boldsymbol{A}(t)$ preserves the translational symmetry, the crystal momentum $\boldsymbol{q}$ is a constant of motion and satisfy

$$\dot{\boldsymbol{q}} = 0. \tag{7}$$

From Eqs. (2) and (5) and we have

$$\dot{\boldsymbol{k}} = -\frac{e}{\hbar}\boldsymbol{E}. \tag{8}$$

*Electronic address: usami@qc.rcast.u-tokyo.ac.jp

Problem 1

By following the similar argument we used in the discussion of *Thouless pumping* [4] and utilizing Eq. (8) show that the velocity of the electron in the eigenstate labeled by $\boldsymbol{k}$ can be given by

$$\boldsymbol{v}_{n,\boldsymbol{k}} = \frac{\partial \epsilon_{n,\boldsymbol{k}}}{\hbar \partial \boldsymbol{k}} \underbrace{- \frac{e}{\hbar}\left(\boldsymbol{E} \times \Omega_{n,\boldsymbol{k}}\right)}_{\boldsymbol{v}^{(1)}_{n,\boldsymbol{k}}}, \tag{9}$$

where the eigen energy $\epsilon_{n,\boldsymbol{k}}$ is given by

$$\langle u_{n,\boldsymbol{k}}|e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} H_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}|u_{n,\boldsymbol{k}}\rangle = \langle u_{n,\boldsymbol{k}}|\epsilon_{n,\boldsymbol{k}}|u_{n,\boldsymbol{k}}\rangle, \tag{10}$$

and the Berry curvature $\Omega_{n,\boldsymbol{k}}$ of the nth band is given by

$$\Omega_{n,\boldsymbol{k}} = i \begin{bmatrix} \left\langle \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_x} \right| \\ \left\langle \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_y} \right| \\ 0 \end{bmatrix} \times \begin{bmatrix} \left| \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_x} \right\rangle \\ \left| \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_y} \right\rangle \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ i \left\langle \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_x} \middle| \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_y} \middle| \frac{\partial u_{n,\boldsymbol{k}}}{\partial k_x} \right\rangle \end{bmatrix}. \tag{11}$$

For hints, please refer to Ref. [1].

The second term in Eq. (9), $\boldsymbol{v}^{(1)}_{n,\boldsymbol{k}}$, is transverse to the electric field $\boldsymbol{E}$ and is called *anomalous velocity*, which is responsible for the *quantum Hall effect* as shown in the next section.

II. THE QUANTUM HALL EFFECT

The Hall current j_1 perpendicular to $\boldsymbol{E}$, which results from the second term in the velocity Eq. (9), can be expressed as

$$j_1 = -e \sum_n \int_{\text{MBZ}} \frac{d\boldsymbol{k}}{(2\pi)^2} \boldsymbol{v}^{(1)}_{n,\boldsymbol{k}} \tag{12}$$

$$= -E \underbrace{\frac{e^2}{\hbar} \sum_n \int_{\text{MBZ}} \frac{d\boldsymbol{k}}{(2\pi)^2} \Omega_{n,\boldsymbol{k}}}_{\sigma_{xy}} \tag{13}$$

where the integration is over the so-called *magnetic Brillouin zone* (MBZ) [1–3]. Since $\sum_n \int_{\text{MBZ}} \frac{d\boldsymbol{k}}{2\pi} \Omega_{n,\boldsymbol{k}}$ can be shown to have some integer value i [1–3], the Hall conductivity σ_{xy} in Eq. (13) can be written by the simple form

$$\sigma_{xy} = \frac{e^2}{h} i \tag{14}$$

and thus be seen to be *quantized* in units of $\frac{e^2}{h}$.

Problem 2

In 1980, von Klitzing, Dorda, and Pepper reported [5] an experiment which measured the quantized Hall *resistance* of the value

$$R_H = 6453.17 \pm 0.02\ \Omega. \tag{15}$$

How can this precise quantized value of the Hall resistance be interpreted in the light of Eq. (14)? How is this value related to the fine structure constant $\alpha \approx \frac{1}{137}$?

[1] D. Xiao, M. -C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
[2] D. J. Thoules, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[3] M. Kohmoto, Ann. Phys. **160**, 343 (1985).
[4] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).
[5] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).