Aharonov-Bohm phase and flux quantization

Koji Usami*

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A charged particle moving in a magnetic field experiences the Lorentz force. Quantum mechanically the charged particle experiences not only the Lorentz force but also the phase shift, which persists even in the space where there is no magnetic field but vector potential. We shall look at this bizarre phase called the *Aharonov-Bohm phase* with Feynman path integral method. This phase is the first example of the *topological terms* in path integrals. The many-body version of Aharonov-Bohm phase is seen in ring-shaped superconductor as the *flux quantization*.

I. VECTOR POTENTIAL [1–3]

Let us start by considering a charged particle in a magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, where $\boldsymbol{A}$ is the vector potential. The starting point is to assume that the probability amplitude that the particle goes from $\boldsymbol{x}$ at time t to $\boldsymbol{x} + \Delta\boldsymbol{x}$ at time $t + \Delta t$ is given by

$$\langle \boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t | \boldsymbol{x}, t \rangle = \langle \boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t | \boldsymbol{x}, t \rangle_{A=0} \underbrace{\exp\left[\frac{i}{\hbar} q\boldsymbol{A}(\boldsymbol{x}) \cdot \Delta\boldsymbol{x}\right]}_{\text{extra phase factor}}. \tag{1}$$

where $\langle \boldsymbol{x} + \Delta\boldsymbol{x}, t + \Delta t | \boldsymbol{x}, t \rangle_{A=0}$ is the same probability amplitude but $\boldsymbol{A}$ is absent. Here Δt and Δx are assumed to be small. Equation (1) states that the particle acquires the extra phase which depends on the vector potential $\boldsymbol{A}$ and the path. With Feynman path integral the above statement means

$$\begin{aligned}
\langle \boldsymbol{x}_f, t_f | \boldsymbol{x}_i, t_i \rangle &= \int_{\boldsymbol{x}_i}^{\boldsymbol{x}_f} D\boldsymbol{x} \exp\left[\frac{i}{\hbar} S[\boldsymbol{x}]\right] \\
&= \int \prod_{k=1}^{N-1} d\boldsymbol{x}_k \exp\left[\frac{i}{\hbar} S[\boldsymbol{x}_k]\right] \\
&= \int \prod_{k=1}^{N-1} d\boldsymbol{x}_k \exp\left[\frac{i}{\hbar} \Delta t \sum_{k=0}^{N-1} \left(L_0(\boldsymbol{x}_k, \dot{\boldsymbol{x}}_k)\right)\right] \underbrace{\exp\left[\frac{i}{\hbar} \sum_{k=0}^{n-1} q\boldsymbol{A}(\boldsymbol{x}_k) \cdot \Delta\boldsymbol{x}_k\right]}_{\text{extra phase factor}} \\
&= \int \prod_{k=1}^{N-1} d\boldsymbol{x}_k \exp\left[\frac{i}{\hbar} \Delta t \sum_{k=0}^{N-1} \left(L_0(\boldsymbol{x}_k, \dot{\boldsymbol{x}}_k)\right)\right] \underbrace{\exp\left[\frac{i}{\hbar} \Delta t \sum_{k=0}^{n-1} q\boldsymbol{A}(\boldsymbol{x}_k) \cdot \dot{\boldsymbol{x}}_k\right]}_{\text{extra phase factor}} \\
&= \int \prod_{k=1}^{N-1} d\boldsymbol{x}_k \exp\left[\frac{i}{\hbar} \Delta t \sum_{k=0}^{N-1} \left\{\left(L_0(\boldsymbol{x}_k, \dot{\boldsymbol{x}}_k)\right) + q\boldsymbol{A}(\boldsymbol{x}_k) \cdot \dot{\boldsymbol{x}}_k\right\}\right] \\
&= \int_{\boldsymbol{x}_i}^{\boldsymbol{x}_f} D\boldsymbol{x} \exp\left[\frac{i}{\hbar} \int_{t_i}^{t_f} \left\{\left(L_0(\boldsymbol{x}, \dot{\boldsymbol{x}})\right) + q\boldsymbol{A}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}}\right\}\right],
\end{aligned} \tag{2}$$

where $L_0(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} m\dot{\boldsymbol{x}}^2$ is the Lagrangian with $\boldsymbol{A}$ being absent. Equation. (2) thus suggests that the action $S[\boldsymbol{x}]$ can be written as

$$\begin{aligned}
S[\boldsymbol{x}] &= S_0[\boldsymbol{x}] + S_{\text{top}}[\boldsymbol{x}] \\
&= \int_{t_i}^{t_f} dt L_0(\boldsymbol{x}, \dot{\boldsymbol{x}}) + q \int_{t_i}^{t_f} dt \boldsymbol{A}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}},
\end{aligned} \tag{3}$$

* usami@qc.rcast.u-tokyo.ac.jp

where $S_0[\boldsymbol{x}]$ is the action with $\boldsymbol{A}$ being absent.

It is reassuring that the Lagrangian for the charged particle in the magnetic field is indeed given by

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2} m \dot{\boldsymbol{x}}^2 + q \boldsymbol{A} \cdot \dot{\boldsymbol{x}}, \tag{4}$$

which also leads to the well-known Hamiltonian by Legendre transformation:

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{p}) &= \dot{\boldsymbol{x}} \boldsymbol{p} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}) \\ &= \frac{1}{2m} (\boldsymbol{p} - q\boldsymbol{A})^2 \\ &= \frac{1}{2m} \Pi^2, \end{aligned} \tag{5}$$

where

$$\boldsymbol{p} = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + q\boldsymbol{A} \tag{6}$$

is the *canonical momentum* and satisfies the standard commutation relation

$$[p_i, p_j] = 0. \tag{7}$$

The statement Eq. (1) is thus the same as that the cannonical momentum $\boldsymbol{p}$ in the free particle should be displaced by $-q\boldsymbol{A}$ in the magnetic field, that is, the famous rule:

$$\boldsymbol{p} = -i\hbar \boldsymbol{\nabla} \to \boldsymbol{p} - q\boldsymbol{A} = -i\hbar \boldsymbol{\nabla} - q\boldsymbol{A}. \tag{8}$$

The shifted momentum

$$\boldsymbol{\Pi} = \boldsymbol{p} - q\boldsymbol{A} = m \frac{d\boldsymbol{x}}{dt} \tag{9}$$

is called the *kinematical momentum*. The kinematical momentum is the *observable (thus gauge-invariant) momentum* but follows the strange commutation relation

$$[\Pi_i, \Pi_j] = i\hbar q \epsilon_{ijk} B_k, \tag{10}$$

as is easily verified with $\boldsymbol{p} = -i\hbar\boldsymbol{\nabla}$, where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is the magnetic field.

II. LORENTZ FORCE

To see that the derived Hamiltonian Eq. (5) from the assumption Eq. (1) is really sensible, let us examine the equation of motion for the charged particle. The Heisenberg equation of motion for the first component of kinematical momentum Π_1 is given by

$$\begin{aligned} m\ddot{x}_1 = \dot{\Pi}_1 &= \frac{i}{\hbar} [H, \Pi_1] \\ &= \frac{i}{2m\hbar} \left[\Pi_1^2 + \Pi_2^2 + \Pi_3^2, \Pi_1\right] \\ &= \frac{q}{2m} \left\{(\Pi_2 B_3 - \Pi_3 B_2) - (B_2 \Pi_3 - B_3 \Pi_2)\right\} \\ &= \frac{q}{2} \left\{(\dot{x}_2 B_3 - \dot{x}_3 B_2) - (B_2 \dot{x}_3 - B_3 \dot{x}_2)\right\} \\ &= \frac{q}{2} \left\{(\dot{\boldsymbol{x}} \times \boldsymbol{B})_1 - (\boldsymbol{B} \times \dot{\boldsymbol{x}})_1\right\} \end{aligned} \tag{11}$$

where the commutation relation Eq. (10) was used. The similar equations are obtained for Π_2 and Π_3 leading to

$$m\ddot{\boldsymbol{x}} = \frac{q}{2} \left(\dot{\boldsymbol{x}} \times \boldsymbol{B} - \boldsymbol{B} \times \dot{\boldsymbol{x}}\right). \tag{12}$$

The right-hand-side of Eq. (12) is indeed the *quantum-mechnical version* of the Lorentz force. The important point here is that there is no gauge-dependent quantities, such as vector potential $\boldsymbol{A}$ and cannonical momentum $\boldsymbol{p}$, appeared in Eq. (12). Thus, even though we start from the manifestly gauge-dependent Lagrangian, Eq. (4), the equation of motion, that is, the observable consequence of the gauge-dependent Lagrangian, turns out to be gauge-invariant!

III. AHARONOV-BOHM PHASE

The above sane and peaceful statement is challenged by Aharonov and Bohm in 1959 [4]. Now suppose that a charged particle moving in a ring on the $xy-$plane as shown in Fig 1. This *not-simply-connected* topology of the ring would turn out to be critical. Here the inter (outer) radius of the ring is ρ_a (ρ_b). It is assumed that there is an uniform magnetic field $\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$ at the center region of the ring up to radius $\rho_0 < \rho_a$ (shown red in Fig 1) but there is no magnetic field at the ring as shown in Fig 1. Thus the particle within the ring would not experience the Lorentz force.

However, what Aharonov and Bohm discovered in 1959 is that quantum mechanically the charged particle experiences the *phase shift*, which persists even in a space where there is no magnetic field but vector potential. Here, we shall see the power of the path integral method by which the extra bizarre phase shift, i.e., the *Aharonov-Bohm phase*, appear in a natural way.

In the cylindrical coordinates the vector potential *in* the ring at radius $\rho \in [\rho_a, \rho_b]$ can be given by

$$\boldsymbol{A} = \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{B\rho_0^2}{2\rho} \\ 0 \end{bmatrix}. \tag{13}$$

This can be verfiied by the Stokes' theorem:

$$\begin{aligned} \int_{\text{disk bouned by radius } \rho} \boldsymbol{B} \cdot d\boldsymbol{S} &= \int_{\text{disk bouned by radius } \rho} (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} \\ &= \oint_{\text{circle of radius } \rho} \boldsymbol{A} \cdot d\boldsymbol{l}. \end{aligned} \tag{14}$$

Here the left-hand-side yields

$$\int_{\text{disk bouned by radius } \rho} \boldsymbol{B} \cdot d\boldsymbol{S} = \pi \rho_0^2 B, \tag{15}$$

while the right-hand-side yields

$$\oint_{\text{circle of radius } \rho} \boldsymbol{A} \cdot d\boldsymbol{l} = 2\pi\rho A_\phi, \tag{16}$$

leading to Eq. (13).

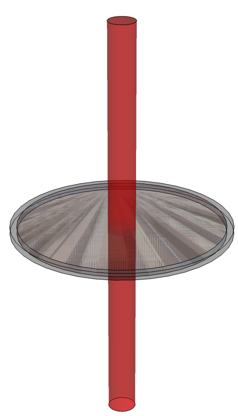

FIG. 1. A ring on the $xy-$plane in which a charged particle moving. The inter (outer) radius of the ring is ρ_a (ρ_b). There is an uniform magnetic field $\boldsymbol{B}$ at the center region of the ring up to radius $\rho_0 < \rho_a$

Let us assume that the radial movement and $z-$axis movement are tightly-constrained and fronzen. The Lagrangian for a particle in this tightly-constrained environment can be given by

$$L_{\rm AB}(\phi,\dot{\phi}) = \frac{m}{2}\left(\rho\dot{\phi}\right)^2 + qA_\phi\left(\rho\dot{\phi}\right), \tag{17}$$

where in the cylindrical coordinate system (r,ϕ,z) the coordinate of the particle can be given by $\rho\phi$ where $\rho\in[\rho_a,\rho_b]$ is the radial coordinate and $\phi\in[0,2\pi]$ is the angular coordinate. The Lagrangian Eq. (17) is thus the same type as the one given in Eq. (4) even though there is no magnetic field at the particle. The extra action $S_{\rm top}[\phi]$ thus becomes

$$\begin{aligned} S_{\rm top}[\phi] &= \int_{t_i}^{t_f} dt\; qA_\phi\left(\rho\frac{d\phi}{dt}\right) \\ &= \int_{\phi_i}^{\phi_f} q\rho A_\phi d\phi = q\rho A_\phi\left[\phi(t_f)-\phi(t_i)\right] = q\Phi\left(\frac{1}{2\pi}\left[\phi_f-\phi_i\right]\right), \end{aligned} \tag{18}$$

where Φ is the magnetic flux threaded at the center region, that is,

$$\Phi = \int_{\text{disk bouned by radius }\rho} \boldsymbol{B}\cdot d\boldsymbol{S} = \pi\rho_0^2 B = 2\pi\rho A_\phi. \tag{19}$$

From Eq. (2) the particle acquires the extra phase factor

$$\exp\left[\frac{i}{\hbar}S_{\rm top}[\phi]\right] = \exp\left[\frac{i}{\hbar}\oint_{\rm ring} q\rho A_\phi d\phi\right] \tag{20}$$

$$= \exp\left[i\frac{q\Phi}{\hbar}\right], \tag{21}$$

when it traverses the ring once, that is, $[\phi_f-\phi_i]=2\pi$. This phase is called *Aharonov-Bohm phase*.

Like the Lorentz force we have seen before, the Aharonov-Bohm phase acquired by the particle traversing the full circle(s) can be *observable* since it can be given by the gauge-invariant quantities Φ, the magnetic flux, according to Eq. (21). However, this magnetic flux is not directly interacting with the particle in the current geometry. Thus, the phase appears as a *nonlocal effect*. Otherwise, we should abandon one of the most fundamental tenets that states that *the observable consequences of the gauge-dependent Lagrangian should be gauge-invariant* and think the guage-dependent formula, Eq. (20), as the more fundamental status than the nonlocal formula but gauge-independent one, Eq. (21).

When the particle traverses the ring n times, the Aharonov-Bohm phase becomes

$$\exp\left[\frac{i}{\hbar}S_{\rm top}[\phi]\right] = \exp\left[i\frac{q\Phi}{\hbar}\right]n. \tag{22}$$

The number n is called the *winding number*.

The situation revolving around the Aharonov-Bohm phase has something to do with the nontrivial *topology* of the Hilbert space where the charged particle lives in the current problem (that is, the ring or circle) as well as the existence of magnetic field. Equations (20), (21), and (22) are the first examples of the *topological terms* in path integrals. We shall learn that the similar terms appear in the different circumstances and dictate the weird but observable behaviors.

IV. FLUX QUANTIZATION

As a final remark on the *topological terms*, we shall look at the relation between the Aharonov-Bohm phase and a quantized phenomenon in the superconducting ring, that is, *flux quantization*. This can be served as a prototype of the topological quantum physics. To see this connection, we shall first study some basic things on the superconductivity.

A. From Schrödinger to London [1]

The Schrödinger equation of a charged particle in an electromagnetic field with a vector potential $\boldsymbol{A}$ can be given by

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) &= H(\boldsymbol{r})\psi(\boldsymbol{r},t) \\ &= \frac{1}{2m}\left(\underbrace{-i\hbar\boldsymbol{\nabla}}_{\boldsymbol{p}}-q\boldsymbol{A}(\boldsymbol{r})\right)\left(\underbrace{-i\hbar\boldsymbol{\nabla}}_{\boldsymbol{p}}-q\boldsymbol{A}(\boldsymbol{r})\right)\psi(\boldsymbol{r},t). \end{aligned} \tag{23}$$

Here note that $\psi(\boldsymbol{r},t)$ is the wave function, $\langle\boldsymbol{r}|\psi(t)\rangle$ and $H(\boldsymbol{r})$ is the position representation of the Hamiltonian operator $\hat{H}$. To be more explicit, we should have written Eq. (23) as

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\langle\boldsymbol{r}|\psi(t)\rangle &= \left\langle\boldsymbol{r}\middle|\hat{H}\middle|\psi(t)\right\rangle \\ &= \int d\boldsymbol{r}'\left\langle\boldsymbol{r}\middle|\hat{H}\middle|\boldsymbol{r}'\right\rangle\langle\boldsymbol{r}'|\psi(t)\rangle \\ &= \int d\boldsymbol{r}' H(\boldsymbol{r}')\underbrace{\langle\boldsymbol{r}|\boldsymbol{r}'\rangle}_{\delta(\boldsymbol{r}'-\boldsymbol{r})}\langle\boldsymbol{r}'|\psi(t)\rangle \\ &= H(\boldsymbol{r})\langle\boldsymbol{r}|\psi(t)\rangle \\ &= \frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla}-q\boldsymbol{A}(\boldsymbol{r})\right)\left(-i\hbar\boldsymbol{\nabla}-q\boldsymbol{A}(\boldsymbol{r})\right)\langle\boldsymbol{r}|\psi(t)\rangle. \end{aligned} \tag{24}$$

Since the probability density $P(\boldsymbol{r},t)$ in quantum mechanics is given in terms of the wave function $\psi(\boldsymbol{r},t)$ by

$$P(\boldsymbol{r},t) = \psi^*(\boldsymbol{r},t)\psi(\boldsymbol{r},t), \tag{25}$$

the *probability current* $\boldsymbol{J}(\boldsymbol{r},t)$ can be obtained from the continuity equation:

$$\frac{\partial}{\partial t}P(\boldsymbol{r},t) = -\nabla\cdot\boldsymbol{J}(\boldsymbol{r},t). \tag{26}$$

where

$$\boldsymbol{J}(\boldsymbol{r},t) = \left[\left(\frac{-i\hbar\nabla-q\boldsymbol{A}(\boldsymbol{r})}{2m}\psi\right)^*\psi+\psi^*\left(\frac{-i\hbar\nabla-q\boldsymbol{A}(\boldsymbol{r})}{2m}\psi\right)\right]. \tag{27}$$

This can be verified by noting that

$$\begin{aligned} \frac{\partial}{\partial t}P(\boldsymbol{r},t) &= \frac{\partial\psi^*(\boldsymbol{r},t)}{\partial t}\psi(\boldsymbol{r},t)+\psi^*(\boldsymbol{r},t)\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} \\ &= \left(-i\frac{H(\boldsymbol{r}')}{\hbar}\psi(\boldsymbol{r},t)\right)^*\psi(\boldsymbol{r},t)+\psi^*(\boldsymbol{r},t)\left(-i\frac{H(\boldsymbol{r}')}{\hbar}\psi(\boldsymbol{r},t)\right) \\ &= \frac{i}{\hbar}\left[\left\{\frac{1}{2m}\left(i\hbar\boldsymbol{\nabla}-q\boldsymbol{A}(\boldsymbol{r})\right)^2\psi^*(\boldsymbol{r},t)\right\}\psi(\boldsymbol{r},t)-\psi^*(\boldsymbol{r},t)\left\{\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla}-q\boldsymbol{A}(\boldsymbol{r})\right)^2\psi(\boldsymbol{r},t)\right\}\right] \\ &= -\boldsymbol{\nabla}\cdot\left[\left(\frac{i\hbar\nabla-q\boldsymbol{A}(\boldsymbol{r})}{2m}\psi^*\right)\psi+\psi^*\left(\frac{-i\hbar\nabla-q\boldsymbol{A}(\boldsymbol{r})}{2m}\psi\right)\right]. \end{aligned} \tag{28}$$

Here is the crucial point: we new consider the wave function $\psi(\boldsymbol{r})$ in Eq. (23) as a *macroscopic* one by identifying it as an *order parameter* of a superconducting metal;

$$\psi(\boldsymbol{r},t) = \sqrt{\rho(\boldsymbol{r})}e^{i\theta(\boldsymbol{r})}, \tag{29}$$

where ρ is the *charge density* and θ is the *phase*. Note the phase θ appears here as a result of the *broken gauge symmetry*. Equation. (27) is now nothing but the *electric current* as opposed to the probability current. The current can be explicitly given in terms of ρ and θ by

$$\boldsymbol{J}(\boldsymbol{r}) = \frac{\hbar}{m}\left(\nabla\theta(\boldsymbol{r})-\frac{q}{\hbar}\boldsymbol{A}(\boldsymbol{r})\right)\rho(\boldsymbol{r}). \tag{30}$$

Without the first term Eq. (30) is the so-called *second London equation* [5];

$$\boldsymbol{J}(\boldsymbol{r}) = -\frac{q}{m}\rho(\boldsymbol{r})\boldsymbol{A}(\boldsymbol{r}), \tag{31}$$

which explains the *perfect conductivity* as well as the *Meissner effect* as follows.

Note that, strictly speaking, Eq. (31) is true only for the type-II (second kind, or London) superconductors (Nb_3Sn, V_3Ga, etc). For the so-called type-I (first kind, or, Pippard) superconductors (Al, Pb, Hg, etc), the relation between $\boldsymbol{J}(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$ becomes a bit more complex and nonlocal [7]. But the following argument can be equally apply to the type-I superconductors when modifying the London penetration depth given by Eq. (44) appropriately [7].

B. Consequences

1. *Perfect conductivity [5]*

From now on, let us assume that the charge density $\rho(\boldsymbol{r})$ in the superconductor is homogeneous, that is, $\rho(\boldsymbol{r}) = \rho$. In ordinary circumstances this is indeed the case. Taking time derivative of Eq. (31), we have

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = -\frac{q\rho}{m}\dot{\boldsymbol{A}}(\boldsymbol{r}). \tag{32}$$

Since $\dot{\boldsymbol{A}}(\boldsymbol{r}) = -\boldsymbol{E}(\boldsymbol{r})$, we have

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = \frac{q\rho}{m}\boldsymbol{E}(\boldsymbol{r}), \tag{33}$$

which suggests the perfect conductivity. The situation is analogous to the lossless free mass system:

$$\underbrace{\dot{\boldsymbol{p}}}_{m\dot{\boldsymbol{J}}(\boldsymbol{r})} = \underbrace{\boldsymbol{F}}_{\rho q\boldsymbol{E}}. \tag{34}$$

There were, on the other hand, a loss term, we have

$$\dot{\boldsymbol{p}} = -\frac{1}{\tau}\boldsymbol{p} + \boldsymbol{F}, \tag{35}$$

and thus

$$\dot{\boldsymbol{J}}(\boldsymbol{r}) = -\frac{1}{\tau}\boldsymbol{J}(\boldsymbol{r}) + \frac{\rho q}{m}\boldsymbol{E}, \tag{36}$$

where τ is the *relaxation time* [6]. Suppse that the τ is very short, we can set $\dot{\boldsymbol{J}}(\boldsymbol{r}) = 0$ in Eq. (36). This gives us

$$\boldsymbol{J}(\boldsymbol{r}) = \underbrace{\frac{\rho q}{m}\tau}_{\sigma}\boldsymbol{E}, \tag{37}$$

and recovering the standard *Ohm's law* within the Drude theory [6].

2. *Meissner effect [1, 5]*

Let us now analyze the consequence of the second London equation (31) upon the magnetic field in a superconductor. Let us start by writing the Maxwell equations

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \tag{38}$$
$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{J}(\boldsymbol{r}) \tag{39}$$

in terms of the vecotor potential $\boldsymbol{A}(\boldsymbol{r})$, the first one becomes obvious and the second one becomes

$$\nabla \times \underbrace{(\nabla \times \boldsymbol{A}(\boldsymbol{r}))}_{\boldsymbol{B}(\boldsymbol{r})} = \mu_0\boldsymbol{J}(\boldsymbol{r}). \tag{40}$$

With the Coulomb gauge $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$ and using vector identity

$$\nabla \times \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \nabla \left(\nabla \cdot \boldsymbol{A}(\boldsymbol{r})\right) - \nabla^2 \boldsymbol{A}(\boldsymbol{r}) \tag{41}$$

Eq. (40) becomes

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = -\mu_0 \boldsymbol{J}(\boldsymbol{r}). \tag{42}$$

Using the second London equation (31) we have a simple equation for $\boldsymbol{A}(\boldsymbol{r})$:

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = \underbrace{\frac{\mu_0 \rho q}{m}}_{\frac{1}{\lambda_L{}^2}} \boldsymbol{A}(\boldsymbol{r}) \to \left(\frac{1}{\lambda_L{}^2} - \nabla^2\right) \boldsymbol{A}(\boldsymbol{r}) = 0, \tag{43}$$

where

$$\lambda_L = \sqrt{\frac{m}{\mu_0 \rho q}} \tag{44}$$

is called the *London penetration depth* [5–7].

From the second London equation Eq. (31) with $\rho(\boldsymbol{r}) = \rho$ we have $\boldsymbol{J}(\boldsymbol{r}) = -\frac{q\rho}{m}\boldsymbol{A}(\boldsymbol{r})$. Thus, Eq (43) also means

$$\left(\frac{1}{\lambda_L{}^2} - \nabla^2\right) \boldsymbol{J}(\boldsymbol{r}) = 0. \tag{45}$$

Moreover, we have one more. By performing operation $\nabla\times$ on the both sides of Eq. (43) we have the similar equation for $\boldsymbol{B}(\boldsymbol{r})$:

$$\left(\frac{1}{\lambda_L{}^2} - \nabla^2\right) \boldsymbol{B}(\boldsymbol{r}) = 0, \tag{46}$$

which is called the *first London equation*.

Let us consider the solution of Eq. (46) for the simple case where the infinitely extended surface of the superconductor lays in the xy plain and the region $z > 0$ being vacuum (this means $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}(z)$). There are two possible cases [7]:

(a): $\boldsymbol{B}(z)$ is parallel to z (that is, $\boldsymbol{B}(z) = \begin{bmatrix} 0 \\ 0 \\ B(z) \end{bmatrix}$)

(b): $\boldsymbol{B}(z)$ is perpendicular to z (say, along x, that is, $\boldsymbol{B}(z) = \begin{bmatrix} B(z) \\ 0 \\ 0 \end{bmatrix}$)

As for the case (a) from Eq. (38) we have $\frac{\partial B}{\partial z} = 0$ and thus $\boldsymbol{B}$ should be spatially constant. This leads to $\nabla \times \boldsymbol{B} = 0$ and thus from Eq. (39) $\boldsymbol{J}(\boldsymbol{r}) = 0$. Therefore it is impossible to have a vector potential $\boldsymbol{A}(\boldsymbol{r})$ normal to the surface of the superconductor.

The case (b) satisfies Eq. (38) automatically. Equation (46) becomes

$$\left(\frac{1}{\lambda_L{}^2} - \frac{\partial^2}{\partial z^2}\right) B(z) = 0. \tag{47}$$

Thus the soltion is

$$B(z) = B_0 \exp\left(-\frac{z}{\lambda_L}\right), \tag{48}$$

suggesting the exponential decay of the magnetic field inside the superconductor, i.e., the *Meissner effect*. The similar conclusion also holds for the current $\boldsymbol{J}(\boldsymbol{r})$ and the vector potential $\boldsymbol{A}(\boldsymbol{r})$, that is, the exponential decay of the current and the vector potential inside the superconductor, since Eqs. (45) and (43) and Eq. (46) are the same.

C. Flux quantization [1]

The existence of the θ term in Eq. (30) produces an even more remarkable phenomenon, *flux quantization.* Let us consider the similar ring shown in Fig. 1, but this time the ring is in the superconducting state. Since the superconducting current flows only near the surface down to the London penetration depth $\lambda_L{}^2$, Eq. (44), the interior current of the ring should be zero. It is emphasized that with the θ term in Eq. (30) the vector potential $\boldsymbol{A}(\boldsymbol{r})$ is not necessarily zero in the inner region of the ring where the current $\boldsymbol{J}(\boldsymbol{r})$ is zero. From Eq. (30) this situation leads to

$$\hbar\nabla\theta(\boldsymbol{r}) = q\boldsymbol{A}(\boldsymbol{r}). \tag{49}$$

Taking the line integral along the path which is within the interior of the ring, we have for the left-hand-side of Eq. (49)

$$\hbar\oint\nabla\theta(\boldsymbol{r})\cdot d\boldsymbol{l} = 2\pi n\,\hbar, \tag{50}$$

to guarantee the *single-valuedness* of the wave function Eq. (29). The line integral of the right-hand-side of Eq. (49) is nothing but the flux:

$$q\oint\boldsymbol{A}(\boldsymbol{r})\cdot d\boldsymbol{l} = q\Phi. \tag{51}$$

But, again, this magnetic flux is acting from distance since there are no magnetic field inside the ring! *The same situation as the one where the Aharonov-Bohm phase appears.* From Eqs. (50) and (51) we reach an another interesting conclusion that the flux Φ has to be *quantized* as

$$\Phi = \frac{2\pi\hbar}{q}n = \frac{h}{q}n \tag{52}$$

with n being any integers $(0,1,2,\cdots)$!

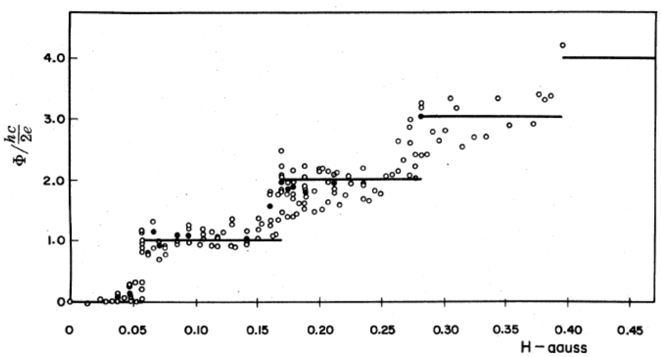

FIG. 2. Experimentally measured *trapped* flux threaded in a superconducting cylinder made out of tin (Sn) as a function of magnetic field, in which the cylinder was cooled below the superconducting transition temperature [8].

In 1961 Deaver and Fairbank [8] as well as Doll and Nabauer [9] experimentally found that q is not the elementary electron charge e but $2e$ (see Fig. 2)! These results reflect the fact that the electrons pairing up as the *Cooper pairs* and being condensed in the ground state (BCS state) in the superconducting metals.

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