Berry phase and Dirac monopole

Koji Usami*

(Dated: October 14, 2021)

We shall explore the similarity between the Lagrangian for a spin-$\frac{1}{2}$ in a magnetic field and that for a charged particle in a magnetic field encountered when we discussed the Aharonov-Bohm phase. We shall then discover the *Berry connection*, the *Berry curvature*, and the *Berry phase* from the former Lagrangian, which correspond to the vector potential, the magnetic field, and the Aharonov-Bohm phase, respectively, appeared from the latter Lagragian. It turns out that the Berry curvature describes the non-zero divergent field associated with a magnetic monopole, called the *Dirac monopole*. We shall also see that the Berry phase appears when a quantum state undergoes an adiabatic evolution with a time-dependent Hamiltonian.

I. BERRY CONNECTION, BERRY CURVATURE, AND BERRY PHASE

We found that the effective Lagrangian for the spin$-\frac{1}{2}$ in the magnetic field $\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$ can be given by

$$\mathcal{L}(\phi, \dot{\phi}, \theta, \dot{\theta}) = \underbrace{\frac{1}{2}\gamma_s B \cos\theta}_{\text{potential energy}} + \underbrace{\frac{i}{2}\left(1-\cos\theta\right)\dot{\phi}}_{\text{velocity dependent part}} , \tag{1}$$

This reminds us of the *imaginary-time* version of the Lagrangian for the charged particle in the vector potential $\boldsymbol{A}$, that is,

$$\mathcal{L}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \underbrace{\frac{1}{2}m\dot{\boldsymbol{x}}^2}_{\text{kinetic energy}} - \underbrace{iq\boldsymbol{A}\cdot\dot{\boldsymbol{x}}}_{\text{velocity dependent part}} , \tag{2}$$

since both Lagrangians contain the *velocity-dependent imaginary parts*. Let us take this analogy seriously and find the corresponding vector potential $\boldsymbol{A}$ for the former.

To this end let us remember that the Euler-Lagrange equation followed from the Lagrangian Eq. (1) is

$$\dot{\boldsymbol{n}} = -\gamma_s \boldsymbol{n} \times \boldsymbol{B}, \tag{3}$$

where the *normalized* magnetic moment $\boldsymbol{n} = \frac{\boldsymbol{m}}{m_0} = -2\boldsymbol{\sigma}$ ($\boldsymbol{m}$: magnetic moment; $m_0 = \frac{g\mu_B}{2}$) can be written in terms of two Euler angles, θ and ϕ, as

$$\boldsymbol{n} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}. \tag{4}$$

Thus, we can consider

$$\dot{\boldsymbol{\sigma}} = -\frac{1}{2}\dot{\boldsymbol{n}} = -\frac{1}{2}\left(\dot{\theta}\boldsymbol{e}_\theta + \sin\theta\dot{\phi}\boldsymbol{e}_\phi\right) = -\frac{1}{2}\begin{bmatrix} 0 \\ \dot{\theta} \\ \sin\theta\dot{\phi} \end{bmatrix}, \tag{5}$$

as the more proper velocity for the spin moving on a sphere with radius of $\frac{1}{2}$, where we use the spherical orthonormal

* usami@qc.rcast.u-tokyo.ac.jp

system

$$\boldsymbol{e}_r = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \tag{6}$$

$$\boldsymbol{e}_\theta = \begin{bmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{bmatrix} \tag{7}$$

$$\boldsymbol{e}_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}. \tag{8}$$

The velocity-dependent imaginary part of the Lagrangian $\mathcal{L}(\phi, \dot{\phi}, \theta, \dot{\theta})$ in Eq. (1) can thus be rewritten in a very similar way as the corresponding part of Eq. (2) as

$$\mathcal{L}_B(\phi, \dot{\phi}, \theta, \dot{\theta}) \equiv \frac{i}{2}\left(1 - \cos\theta\right)\dot{\phi} = -i\boldsymbol{A}_\uparrow \cdot \dot{\boldsymbol{\sigma}}, \tag{9}$$

where we defined the vector-potential-like quantity $\boldsymbol{A}_\uparrow$ as

$$\boldsymbol{A}_\uparrow = \begin{bmatrix} 0 \\ 0 \\ \frac{1-\cos\theta}{\sin\theta} \end{bmatrix} \tag{10}$$

in the spherical coordinates. This vector potential is called the *Berry connection* in the literature [1]. The subscript $\uparrow$ emphasizes the fact that the Berry connection stems from the *state* $|\uparrow\rangle$, which we shall see more later on.

The *Berry phase action* can thus be written in three different ways:

$$S_{\text{top}}[\phi, \theta] = -\int_0^\beta d\tau \left\langle \frac{\partial}{\partial\tau} g \middle| g \right\rangle \tag{11}$$

$$= \frac{i}{2}\int_0^\beta d\tau \left(1 - \cos\theta\right)\dot{\phi} \tag{12}$$

$$= -i\int_0^\beta d\tau \boldsymbol{A}_\uparrow \cdot \dot{\boldsymbol{\sigma}}. \tag{13}$$

Now we shall find the another expression of the Berry connection $\boldsymbol{A}_\uparrow$. First, notice that

$$\frac{\partial}{\partial\tau}\langle g|g\rangle = \left\langle \frac{\partial}{\partial\tau} g \middle| g \right\rangle + \left\langle g \middle| \frac{\partial}{\partial\tau} g \right\rangle = 0 \tag{14}$$

and thus

$$\left\langle \frac{\partial}{\partial\tau} g \middle| g \right\rangle = -\left\langle g \middle| \frac{\partial}{\partial\tau} g \right\rangle \tag{15}$$

and $\langle g|\frac{\partial}{\partial\tau} g\rangle$ is pure imaginary. Equation. (11) can thus be rewritten as

$$S_{\text{top}}[\phi, \theta] = \int_0^\beta d\tau \left\langle g \middle| \frac{\partial}{\partial\tau} g \right\rangle. \tag{16}$$

Next, notice $g(\tau)$, a function of τ, can also be viewed as $g(\boldsymbol{\sigma}(\tau))$, a function of $\boldsymbol{\sigma}(\tau)$, that is,

$$S_{\text{top}}[\phi, \theta] = \int_0^\beta d\tau \left\langle g(\boldsymbol{\sigma}(\tau)) \middle| \frac{\partial}{\partial\boldsymbol{\sigma}(\tau)} g(\boldsymbol{\sigma}(\tau)) \right\rangle \dot{\boldsymbol{\sigma}}(\tau). \tag{17}$$

Comparing this expression with Eq. (13) we find the more famous expression of the Berry connection:

$$\boldsymbol{A}_\uparrow = i\left\langle g(\boldsymbol{\sigma}) \middle| \frac{\partial}{\partial\boldsymbol{\sigma}} \middle| g(\boldsymbol{\sigma}) \right\rangle, \tag{18}$$

which, from Eq. (15), is a real-valued quantity.

Let us find some more. Since the integration with respect to τ in Eq.(13) running from $\tau = 0$ to β is traded for the contour integration with respect to $\boldsymbol{\sigma}$, the Berry phase action can be further modified to

$$\begin{aligned} S_{\rm top}[\phi,\theta] &= -i \oint_{\mathcal{C}} \boldsymbol{A}_{\uparrow} \cdot d\boldsymbol{\sigma} \\ &= -i \int_{\mathcal{A}} \underbrace{\left(\boldsymbol{\nabla} \times \boldsymbol{A}_{\uparrow}\right)}_{\boldsymbol{\Omega}_{\uparrow}} \cdot d\boldsymbol{S}, \end{aligned} \tag{19}$$

using Stokes theorem, where $\int_C d\boldsymbol{\sigma}$ is the contour integral with respect to $\boldsymbol{\sigma}$ along the circle $\mathcal{C}$ while $\int_{\mathcal{A}} d\boldsymbol{S}$ is the surface integral over the area $\mathcal{A}$ bounded by the circle $\mathcal{C}$. Here,

$$\boldsymbol{\Omega}_{\uparrow} = \boldsymbol{\nabla} \times \boldsymbol{A}_{\uparrow} \tag{20}$$

is like magnetic field and is called the *Berry curvature* in the literature [1].

Like a charged particle moving in a ring, which is threaded by a magnetic field $\boldsymbol{B}$, acquires the Aharonov-Bohm phase, the magnetic moment moving on the sphere with the Berry curvature $\boldsymbol{\Omega}_{\uparrow}$ acquires the *Berry phase* $\gamma_{\uparrow}$, which is defined by

$$\gamma_{\uparrow} = \oint_{\mathcal{C}} d\boldsymbol{\sigma} \cdot \boldsymbol{A}_{\uparrow} = \int_{\mathcal{A}} d\boldsymbol{S} \cdot \boldsymbol{\Omega}_{\uparrow}. \tag{21}$$

We have thus the following correspondences:

$$\begin{array}{lcl} \text{vector potential} : \boldsymbol{A} & \Leftrightarrow & \text{Berry connection} : \boldsymbol{A}_{\uparrow} \\ \text{magnetic field} : \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} & \Leftrightarrow & \text{Berry curvature} : \boldsymbol{\Omega}_{\uparrow} = \boldsymbol{\nabla} \times \boldsymbol{A}_{\uparrow} \\ \text{Aharonov} - \text{Bohm phase} : \gamma & \Leftrightarrow & \text{Berry phase} : \gamma_{\uparrow} \end{array}.$$

II. DIRAC MONOPOLE

A. Dirac monopole [2]

Now let us explore the Berry connection $\boldsymbol{A}_{\uparrow}$ and the Berry curvature $\boldsymbol{\Omega}_{\uparrow}$ a little bit more. According to the above argument, the Berry curvature $\boldsymbol{\Omega}_{\uparrow}$ is like magnetic field. Then what kind of magnetic field? Taking rotation of $\boldsymbol{A}_{\uparrow}$ given by Eq. (10) in the spherical coordinate system $(r = \frac{1}{2}, \theta, \phi)$ we have

$$\begin{aligned} \boldsymbol{\Omega}_{\uparrow} &= \boldsymbol{\nabla} \times \boldsymbol{A}_{\uparrow} \\ &= \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_{\theta} \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_{\phi} \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\right) \times \left(A_r \boldsymbol{e}_r + A_{\theta} \boldsymbol{e}_{\theta} + A_{\phi} \boldsymbol{e}_{\phi}\right) \\ &= \begin{bmatrix} \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta A_{\phi}\right) - \frac{\partial}{\partial\phi} A_{\theta}\right\} \\ \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} A_r - \frac{\partial}{\partial r}\left(r A_{\phi}\right)\right\} \\ \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r A_{\theta}\right) - \frac{\partial}{\partial\theta} A_r\right\} \end{bmatrix} \end{aligned} \tag{22}$$

$$\begin{aligned} &= \begin{bmatrix} \frac{1}{\frac{1}{2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_{\phi}\right) \\ -\frac{1}{\frac{1}{2}}\frac{\partial}{\partial r}\left(\frac{1}{2} A_{\phi}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \\ &= 2\boldsymbol{e}_r, \end{aligned} \tag{23}$$

where we use $\boldsymbol{\nabla}$ in the spherical coordinate system (see Appendix A for the derivation),

$$\boldsymbol{\nabla} = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_{\theta} \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_{\phi} \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}, \tag{24}$$

and the following relations:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \tag{25}$$

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k \tag{26}$$

$$\frac{\partial}{\partial r}\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

$$\frac{\partial}{\partial \theta}\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_\theta \\ -\boldsymbol{e}_r \\ 0 \end{bmatrix} \tag{28}$$

$$\frac{\partial}{\partial \phi}\begin{bmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \boldsymbol{e}_\phi \\ \cos\theta \boldsymbol{e}_\phi \\ -\sin\theta \boldsymbol{e}_r - \cos\theta \boldsymbol{e}_\theta \end{bmatrix}. \tag{29}$$

Notice that Eq. (23) suggests that the magnetic field is pointing radially, just like the *magnetic monopole*! This monopole is called the *Dirac monopole* in some literature [3]. The strange point of the Dirac monopole is that the divergence of the field $\boldsymbol{\Omega}_\uparrow$ is not zero. This means that the Berry connection $\boldsymbol{A}_\uparrow$ should be ill-behaved since $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}_\uparrow) \neq 0$. This is indeed true. $A_\phi = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$ is singular at $\theta = \pi$ as seen in Fig. 1.

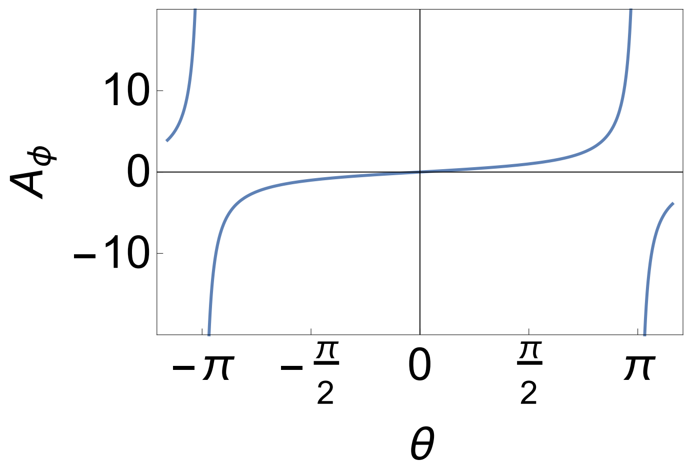

FIG. 1. ϕ-component of Berry connection $\boldsymbol{A}_\uparrow$, $A_\phi = \frac{1-\cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$, as a function of θ.

We could partly remedy this situation by using the other Berry connection, for instance,

$$\boldsymbol{A}_\downarrow = \begin{bmatrix} 0 \\ 0 \\ -\frac{1+\cos\theta}{\sin\theta} \end{bmatrix}, \tag{30}$$

which can be obtained by the following gauge transformation:

$$\begin{aligned} \boldsymbol{A}_\downarrow &= \boldsymbol{A}_\uparrow - \boldsymbol{\nabla}\phi \\ &= \boldsymbol{A}_\uparrow - \left(\boldsymbol{e}_r\frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{\frac{1}{2}}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{\frac{1}{2}\sin\theta}\frac{\partial}{\partial \phi}\right)\phi \\ &= \begin{bmatrix} 0 \\ 0 \\ \frac{1-\cos\theta}{\sin\theta} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\frac{1}{2}\sin\theta} \end{bmatrix}. \end{aligned} \tag{31}$$

Like vector potentials, the Berry connection is thus gauge-dependent. The gauge-transformed Berry connection $\boldsymbol{A}_\downarrow$ in Eq. (30) does not have singularity at $\theta = \pi$, but does have it at $\theta = 0$ as seen in Fig. 2. Note that the Berry connection $\boldsymbol{A}_\downarrow$ produces exactly the same Berry curvature $\boldsymbol{\Omega}_\downarrow$ as $\boldsymbol{\Omega}_\uparrow$ in Eq. (23), thus, like magnetic field, the Berry curvature is gauge-independent.

What about the Berry phase $\gamma_\uparrow$ in Eq. (21)? Does it change by the gauge transformation Eq. (31)? Let us see the interesting answer to this question. Remember that $\boldsymbol{\sigma}$ in Eq. (21) traverses the circle $\mathcal{C}$ on the sphere of radius $\frac{1}{2}$. Let

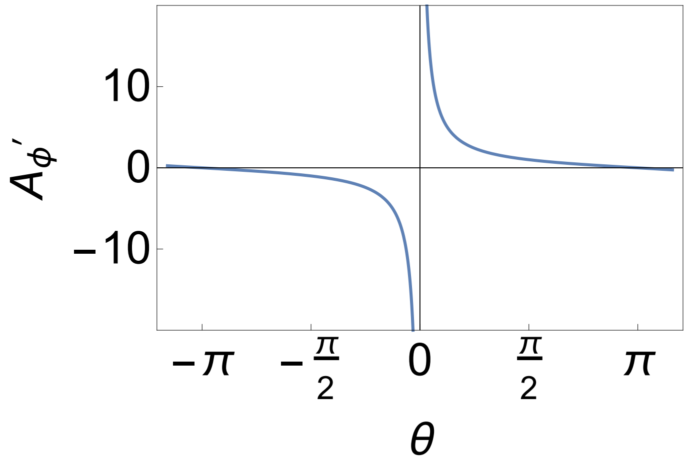

FIG. 2. ϕ-component of Berry connection $\boldsymbol{A}_\downarrow$, $A'_\phi = -\frac{1+\cos\theta}{\sin\theta} = \frac{1}{\tan\frac{\theta}{2}}$, as a function of θ.

us suppose the area $\mathcal{A}$ enclosed by $\mathcal{C}$ is $A_{\mathcal{C},\uparrow}$ when the area contains the north pole and $A_{\mathcal{C},\downarrow}$ when the area contains the south pole. The Berry phase can then be given as

$$\gamma_\uparrow = \oint_{\mathcal{C}} d\boldsymbol{l}\cdot\boldsymbol{A}_\uparrow = \int_{\mathcal{A}_{\mathcal{C},\uparrow}} d\boldsymbol{S}\cdot \underbrace{\boldsymbol{\Omega}_\uparrow}_{\boldsymbol{\nabla}\times\boldsymbol{A}_\uparrow} = \int_{\mathcal{A}_{\mathcal{C},\uparrow}} d\boldsymbol{S}\cdot 2\boldsymbol{e}_r = 2A_{\mathcal{C},\uparrow}, \tag{32}$$

where $\boldsymbol{A}_\uparrow$ is well-defined within the area $A_{\mathcal{C},\uparrow}$. The same Berry phase can also be expressed in terms of $\boldsymbol{A}_\downarrow$ as

$$\gamma_\downarrow = \oint_{\mathcal{C}} d\boldsymbol{l}\cdot\boldsymbol{A}_\downarrow = \int_{\mathcal{A}_{\mathcal{C},\downarrow}} d\boldsymbol{S}\cdot \underbrace{\boldsymbol{\Omega}_\downarrow}_{\boldsymbol{\nabla}\times\boldsymbol{A}_\downarrow} = \int_{\mathcal{A}_{\mathcal{C},\downarrow}} d\boldsymbol{S}\cdot 2\boldsymbol{e}_r = -2A_{\mathcal{C},\downarrow}, \tag{33}$$

where $\boldsymbol{A}_\downarrow$ is well-defined within the area $A_{\mathcal{C},\downarrow}$. Here, the minus sign comes from the fact that the area here has the *orientation* with respect to the circle $\mathcal{C}$.

Are these two expressions different? To see this, let us calculate the difference:

$$\gamma_\uparrow - \gamma_\downarrow = 2A_{\mathcal{C},\uparrow} + 2A_{\mathcal{C},\downarrow} = 2 \underbrace{4\pi\left(\frac{1}{2}\right)^2}_{\text{sphere surface of radius}\frac{1}{2}} = 2\pi. \tag{34}$$

Thus the answer is no in a sense of modulo 2π! We thus say that $\gamma_\uparrow = \gamma_\downarrow$ and the the Berry phase is gauge-independent!

B. Quntization of spin [2]

We can repeat the similar arguments for the general spin-S cases to reach the conclusion that the Berry phase acquired by the spin-S moving on sphere of radius S is

$$\gamma_\uparrow = \frac{1}{S}A_{\mathcal{C},\uparrow} \tag{35}$$

for the calcuration based on the area contains the north pole while

$$\gamma_\downarrow = -\frac{1}{S}A_{\mathcal{C},\downarrow} \tag{36}$$

for that based on the area contains the south pole. The difference is thus given by

$$\gamma_\uparrow - \gamma_\downarrow = \frac{1}{S}A_{\mathcal{C},\uparrow} + \frac{1}{S}A_{\mathcal{C},\downarrow} = \frac{1}{S} \underbrace{4\pi S^2}_{\text{sphere surface of radius}S} = 4\pi S. \tag{37}$$

We can thus draw a very interesting conclusion that as far as the spin is quantized as $\frac{1}{2}, 1, \frac{3}{2}, \cdots$, the Berry phase can be single-valued (modulo 2π) and gauge-independent. It can also be seen that the minimum possible spin is not 1 but $\frac{1}{2}$!

This in turn means that *the spin has to be quantized if we require that the Berry phase is single-valued (modulo 2π)*! Here, we found a yet another item of correspondence, namely,

$$\text{Flux quantization} \;\Leftrightarrow\; \text{Spin quantization}\,.$$

III. BERRY PHASE AND ADIABATIC CHANGES OF A QUANTUM STATE [1–4]

So far we investigated the Berry phase with path integral method, which basically means that we treated the inherently quantum-mechanical electron spin as the *classical* magnetic moment, $\boldsymbol{n} = \frac{\boldsymbol{m}}{m_0} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$. Now, we shall revisit the Berry phase by analyzing the adiabatic evolution of a *quantum state* $|\Uparrow(t)\rangle$, which is the lowest energy eigenstate of a time-dependent Hamiltonian $H(t)$.

A. Adiabatic changes of a quantum state

Let the time-dependent Hamiltonian be

$$H(t) = -\boldsymbol{m}\cdot\boldsymbol{B}(t) = \hbar\gamma_s\boldsymbol{\sigma}\cdot\boldsymbol{B}(t). \tag{38}$$

Suppose that the magnetic field at $t=0$ is $\boldsymbol{B}(0) = B(0)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and the spin starts at $t=0$ in one of the eigenstates

$$\underbrace{|\Uparrow(0)\rangle}_{\text{for magnetic moment}} = \underbrace{|\Downarrow(0)\rangle}_{\text{for spin}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{39}$$

with the energy $\epsilon_{\Uparrow}(0) = \epsilon_{\Downarrow}(0) = -\frac{1}{2}\hbar\gamma_s B(0)$. When the time-variation of the Hamiltonian $H(t)$ is *adiabatic* the spin state remains in the instantaneous eigenstate of $H(t)$, that is,

$$|\Uparrow(t)\rangle = |\Downarrow(t)\rangle = \begin{bmatrix} -e^{-i\frac{\phi(t)}{2}}\sin\frac{\theta(t)}{2} \\ e^{i\frac{\phi(t)}{2}}\cos\frac{\theta(t)}{2} \end{bmatrix}, \tag{40}$$

with the energy $\epsilon_{\Uparrow}(t) = \epsilon_{\Downarrow}(t) = -\frac{1}{2}\hbar\gamma_s B(t)$. Here, at t the magnetic field is assumed to be

$$\boldsymbol{B}(t) = B(t)\begin{bmatrix} \sin\theta(t)\sin\phi(t) \\ \sin\theta(t)\cos\phi(t) \\ \cos\theta(t) \end{bmatrix}. \tag{41}$$

Now suppose that, at the end of the evolution $t = T$, the Hamiltonian returns to the original one, that is, $H(T) = H(0)$ and thus the state must come back to the original state with some phase factor, that is,

$$|\Uparrow(T)\rangle = e^{-i\Phi(T)}|\Uparrow(0)\rangle. \tag{42}$$

We shall see that the phase can be written as [5]

$$\Phi(T) = \underbrace{\Phi(0)}_{\text{initial phase}} + \underbrace{\frac{1}{\hbar}\int_0^T dt\epsilon_{\Uparrow}(t)}_{\text{dynamical phase}} - \underbrace{\gamma_{\Uparrow}}_{\text{Berry phase}}. \tag{43}$$

Let us start by considering the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle, \tag{44}$$

where the wave function $|\psi(t)\rangle$ can be assumed to be the instantaneous eigenstate $|\uparrow(t)\rangle$ with some phase factor, that is,

$$|\psi(t)\rangle = e^{-i\Phi(t)}|\uparrow(t)\rangle \tag{45}$$

since $|\psi(t)\rangle$ changes adiabatically from $|\uparrow(0)\rangle$ to $|\uparrow(T)\rangle$ in a course of time evolution. This adiabatic approximation is essentially equivalent to performing a *projection operation* on the state $|\psi(t)\rangle$ to restrict it to the eigenstates $|\uparrow(t)\rangle$ [1]. Plugging this form of wave function into Eq. (44) and operate $\langle\uparrow(t)|$ from the left we have

$$\hbar\frac{\partial\Phi(t)}{\partial t} + i\hbar\left\langle\uparrow(t)\left|\frac{\partial}{\partial t}\right|\uparrow(t)\right\rangle = \epsilon_\uparrow(t). \tag{46}$$

By integrating both sides with respect to t from 0 to T we have

$$\hbar\left(\Phi(T)-\Phi(0)\right) + \hbar\int_0^T dt\; i\left\langle\uparrow(t)\left|\frac{\partial}{\partial t}\right|\uparrow(t)\right\rangle = \int_0^T dt\epsilon_\uparrow(t), \tag{47}$$

which indeed indicates Eq.(43) with the Berry phase [5]:

$$\begin{aligned}
\gamma_\uparrow &= \int_0^T dt\; i\left\langle\uparrow(t)\left|\frac{\partial}{\partial t}\right|\uparrow(t)\right\rangle \\
&= \int_0^T dt\left(i\left\langle\uparrow(\boldsymbol{\sigma}(t))\left|\frac{\partial}{\partial\boldsymbol{\sigma}(t)}\right|\uparrow(\boldsymbol{\sigma}(t))\right\rangle\right)\dot{\boldsymbol{\sigma}}(t) \\
&= \oint_{\mathrm{C}} d\boldsymbol{\sigma}\cdot\underbrace{\left(i\left\langle\uparrow(\boldsymbol{\sigma})\left|\frac{\partial}{\partial\boldsymbol{\sigma}}\right|\uparrow(\boldsymbol{\sigma})\right\rangle\right)}_{\boldsymbol{A}_\uparrow:\ \text{Berry connection}} \\
&= \int_{\mathcal{A}} d\boldsymbol{S}\cdot\underbrace{(\boldsymbol{\nabla}\times\boldsymbol{A}_\uparrow)}_{\boldsymbol{\Omega}_\uparrow:\ \text{Berry curvature}}\quad.
\end{aligned} \tag{48}$$

This establishes the close link between the Berry phase and adiabatic evolution of the quantum state $|\uparrow(t)\rangle$. Note that $\gamma_\uparrow$ does not depend on the velocity $\dot{\boldsymbol{\sigma}}$ in this setting and stems from the *geometry* of the space where the eigenstates $|\uparrow(t)\rangle$ lives. Thus, the Berry phase is also called the *geometric phase*.

Appendix A: $\boldsymbol{\nabla}$ in the spherical coordinate system

In the spherical coordinate system, we have

$$x = r\sin\theta\cos\phi \tag{A1}$$
$$y = r\sin\theta\cos\phi \tag{A2}$$
$$z = r\cos\theta. \tag{A3}$$

This leads to the following relationship between (dx, dy, dz) and $(dr, d\theta, d\phi)$:

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}\begin{bmatrix} dr \\ d\theta \\ d\phi \end{bmatrix}. \tag{A4}$$

This, in turn, brings us to

$$\boldsymbol{\nabla} = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial \theta}{\partial y} & \dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial r}{\partial z} & \dfrac{\partial \theta}{\partial z} & \dfrac{\partial \phi}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \dfrac{1}{r}\cos\theta\cos\phi & -\dfrac{1}{r\sin\theta}\sin\phi \\ \sin\theta\sin\phi & \dfrac{1}{r}\cos\theta\sin\phi & \dfrac{1}{r\sin\theta}\cos\phi \\ \cos\theta & -\dfrac{1}{r}\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial \phi} \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi} \end{bmatrix}$$

$$= [\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi] \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi} \end{bmatrix}$$

$$= \left(\boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right). \tag{A5}$$

[1] D. Xiao, M. -C. Chang, and Q. Niu, Rev. Mod. Phys. **82**, 1959 (2010).
[2] A. Altland and B. D. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, Cambridge, 2010).
[3] J. J. Sakurai, *Modern Quantum Mechanics*, revised ed. (Addison-Wesley, Reading, MA, 1994).
[4] D. Vanderbilt, *Berry Phase in Electronic Structure Threory* (Cambridge University Press, Cambridge, 2018).
[5] M. V. Berry, Proc. R. Soc. Lond. A **392**, 45 (1984).